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MEASUREMENT OF SPIN-LATTICE RELAXATION TIME IN Mn^{++} *SALT SOLUTIONS*

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A Q-meter was used to measure the spin-lattice relaxation time and internal field parameters in certain Mn^{++} salt solutions. The spin-lattice relaxation time of $MnCl_2 \cdot 4H_2O$ in glycerin and water-glycerin solutions is satisfactorily described by the Casimir-Du Pré theory when allowance is made for spin-spin absorption, and is also in good agreement with the Brons-Van Vleck formula. The spin-lattice relaxation time is only slightly dependent on the paramagnetic salt concentration and on the viscosity of the solution.

1. APPARATUS AND METHOD OF MEASUREMENT

RELAXATION processes in paramagnetic substances can be investigated by measuring either the high-frequency susceptibility χ' or the absorption component χ''.[1,2] In the present work measurements of χ'' are obtained based on variations in the Q of the coil of the tuned oscillatory circuit. The relation between ΔQ and its cause χ'' is given by[3-5]

$$\Delta Q = -4\pi\eta\chi'' Q^2. \qquad (1)$$

Here Q is the figure of merit and η is the "filling" factor of the coil when occupied by the paramagnet. Equation (1) can be written more conveniently as[3,4]

$$\Delta E = -4\pi\eta\chi'' EQ, \qquad (2)$$

where E is the voltage generated in the tuned circuit and ΔE is the variation of this voltage associated with χ''.

Apparatus which measures Q can be used in two different ways to determine the spin-lattice relaxation time ρ_L in a paramagnetic: 1) by using measurements of χ'' (in absolute units according to (2) or in comparable relative units) over a broad range of frequencies ν to construct $\chi''(\nu)$ curves for a given constant magnetic field H; 2) from measurements of χ'' at two frequencies using the equation of Casimir and Du Pré:[1]

$$\chi'' = \chi_0 F \rho_L \nu / (1 + \rho_L^2 \nu^2), \qquad F = H^2/(b/c + H^2). \qquad (3)$$

Here b/c is a constant which characterizes the internal magnetic field of the paramagnetic.

The basis of the two-frequency method of determining ρ_L is as follows. Writing (3) for a given constant magnetic field H_1 and using two values of the frequency ν, we obtain from (2) and (3)*

$$\rho_{H_1} = \sqrt{(k-1)/(\nu_2^2 - k\nu_1^2)},$$
$$k = \Delta E_{\nu_1} E_{\nu_2} Q_{\nu_2} \nu_2 / \Delta E_{\nu_2} E_{\nu_1} Q_{\nu_1} \nu_1. \qquad (4)$$

Here all quantities that determine ρ_{H_1} can be measured. The equation contains neither η nor the absolute value of χ'', while only the first power of Q appears; the relaxation time is consequently measured more accurately.

It has been verified experimentally that a Q-meter can be used to determine spin-lattice relaxation times from measurements at two frequencies

*Yu. Ya. Shamonin was the first to point out that a commercial type KV-1 Q-meter could be used for oscilloscopic observation of paramagnetic resonance absorption curves.

No. of experiment	H, Oe	1200	1600	2000	2400	2800	3200	3600
1	$\rho_L \cdot 10^8$	1.1	1.45	1.7	2.1	2.45	2.7	3.1
2	$\rho_L \cdot 10^8$	1.4	1.7	1.95	2.3	2.7	2.9	3.15
3	$\rho_L \cdot 10^8$	1.4	1.65	1.85	2.15	2.5	2.75	3.05

according to (4). A Type KV-1 Q-meter fed from lead storage batteries was used to measure Q at several frequencies when a given paramagnet was within the coil. The coil and specimen of $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$ were placed in a static magnetic field which could be directed at will either parallel ($H_\parallel$) or perpendicular ($H_\perp$) to the coil axis. ΔE was measured in the fields $H_\parallel$ and $H_\perp$ at the same frequencies, a PPTV-1 potentiometer being used. Experiments with $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$ at frequencies of 1, 1.5, 2 and 3 Mcs were repeated several times. ρ_L was calculated from the values of Q and ΔE in the field $H_\parallel$ for each of the two frequencies; five values of ρ_L obtained in this way were averaged. The maximum deviation of ρ_L from its mean value was 3% and the results agreed to within 4% with Gorter's data.[1]

Another manganese salt, $MnCl_2 \cdot 4H_2O$, was used in similar experiments at 3.5, 5.25, 7 and 10 Mcs. In this case the relaxation times agree almost as well as Kozyrev's results for paramagnetic absorption[6] (5% deviation from the mean) and not quite so well with Gorter's results[1] for the dispersion of susceptibility. The values obtained for ρ_L were used to calculate b/c; good agreement was obtained with other writers[1,7,8] for both of the salts. These experiments on $Mn(NH_4)_2(SO_4)_2 \cdot 6H_2O$ and $MnCl_2 \cdot 4H_2O$ prove that it is possible to determine spin-lattice relaxation times in paramagnetic salts from (4), by means of the Q-meter, to within ~ 5%.

2. MEASUREMENTS ON GLYCERIN SOLUTIONS OF Mn^{++} SALTS

We also performed experiments to determine the possibility of applying the foregoing method to measure spin-lattice relaxation times and b/c in liquid solutions of paramagnetic salts.

Paramagnetic relaxation in liquid solutions of manganese salts was discovered by Zavoĭskiĭ in 1944.[2*] The literature contains no further information on work of this type, although the study of paramagnetic relaxation times in electrolytic solutions is of considerable interest both for the theory of relaxation processes and for investigations of the structure and molecular properties of solutions. We have therefore attempted to apply the procedure developed for solids to such solutions as well.

The investigation of paramagnetic absorption in electrolytic solutions is rendered difficult by large electric losses which exceed paramagnetic absorption by several orders of magnitude. We may expect, however, that the accuracy of χ'' (and thus of ρ_L) will be independent of electric losses in the Q-meter method, where ΔQ means the change of the coil figure of merit as a result of paramagnetic absorption alone and Q is taken when the electrolyte is within the coil.

Glycerin solutions of $MnCl_2 \cdot 4H_2O$ were used in these experiments at 12, 21.5, 32 and 43 Mcs. Just as in the case of solid manganese salts the comparison method at selected frequencies was used to determine the Q of a coil around the solution contained in a cylindrical glass ampule. The dependence of ΔE on $H_\parallel$ and $H_\perp$ was also obtained. The $\Delta E(H_\perp)$ curve was required in order to determine absorption at $H = 0$ and thus compute the loss resulting from the spin-spin interaction.[1,10] Ampules of different lengths and diameters were used accompanied, of course, by different changes of Q resulting from electric losses.

The table gives the spin-lattice relaxation times calculated from (4) in three experiments on a glycerin solution of $MnCl_2 \cdot 4H_2O$ (3.9 mole/liter). In the first two experiments, conducted under identical conditions, an ampule of 7 mm diameter and 4 mm length was used; at 43 Mcs, Q was reduced from 217 to 210 when the ampule was inserted into the coil. In the third experiment an ampule of 10 mm diameter and 19 mm length was used; Q dropped from 217 to 154 at 43 Mcs.

The large (~ 10%) spread of ρ_L in low fields H resulted from the smallness of the observed effect in such fields. Measurements and calculations for all solutions were performed in the same order as for the solid salts.

The values obtained for ΔE can be checked somewhat roughly by using the formula $\chi''_\perp(0)\Delta H/\chi_0 = \text{const}$, where ΔH is the half-width of the $\chi''(H_\perp)$ curve and $\chi''_\perp(0)$ is the absorption factor for $H_\perp = 0$. This equation has frequently been tested on solid paramagnets.[11] For our measure-

*A similar attempt by Teunisson and Gorter[9] in 1940 was unsuccessful.

ments on glycerin solutions it is accurate to within ~ 10%.

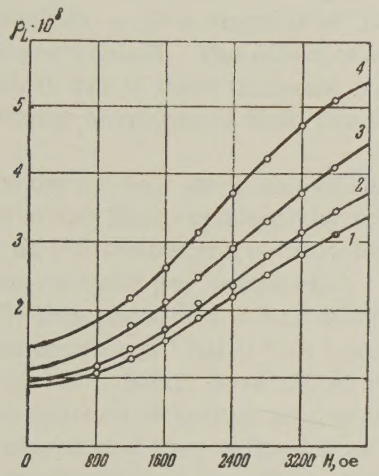
The example described above indicates that it is possible to use the procedure in question for measurements of spin-lattice relaxation times in liquid solutions of paramagnets with an accuracy of ~ 5%.

Glycerin solutions of $MnCl_2 \cdot 4H_2O$ were investigated at concentrations of 3.9, 1.8, 0.9, 0.45 and 0.225 mole/liter. The relaxation times ρ_L were used to determine b/c for all concentrations by means of the formula

$$b/c = (H_2^2 - AH_1^2)/(A-1),$$

$$A = \Delta E_{H_1} H_2^2 \rho_{LH_2} (1 + \rho_{LH_1}^2 \nu^2) / \Delta E_{H_2} H_1^2 \rho_{LH_1} (1 + \rho_{LH_2}^2 \nu^2),$$

which is easily obtained from the spin-lattice relaxation theory.[1] The spread in the values of b/c was 7 — 10%.



Curve 1 — C = 3.9 mole/liter; b/c = 2.5×10^6 oe^2; $\rho_0 = 0.88 \times 10^{-8}$ sec; p = 0.147 Curve 2 — C = 1.8 mole/liter; b/c = 2.4×10^6 oe^2; $\rho_0 = 0.95 \times 10^{-8}$ sec; p = 0.15 Curve 3 — C = 0.9 mole/liter; b/c = 2.27×10^6 oe^2; $\rho_0 = 1.14 \times 10^{-8}$ sec; p = 0.155 Curve 4 — C = 0.45 mole/liter; b/c = 2.1×10^6 oe; $\rho_0 = 1.5 \times 10^{-8}$ sec; p = 0.17

The figure compares the experimental values of $\rho(H)$ (points on the curve) with the theoretical (continuous) curves of $\rho_L(H)$ plotted according to the Brons-Van Vleck equation for all of the investigated concentrations:

$$\rho_L = \rho_0 \frac{b/c + H^2}{b/c + pH^2},$$

where ρ_0 is the spin-lattice relaxation time for H = 0 and p is a constant. The experimental value of ρ_L for concentration c = 0.225 mole/liter does not differ within the limit of error from that for c = 0.45 mole/liter.

An analysis of the experimental values of ρ_L given in the figure shows that the theory of Casimir and Du Pré correctly describes paramagnetic relaxation in glycerin solutions of manganese and that the Brons-Van Vleck equation can be used for such solutions.

The results obtained with solutions of $MnCl_2 \cdot 4H_2O$ indicate that it is possible to investigate the dependence of ρ_L and b/c on the composition of the solvent (water-glycerin mixture) and on the amount of diamagnetic salt that is added. The first of such experiments determines the dependence of ρ_L and b/c on the viscosity of the solutions; the second experiment determines the dependence of the same quantities on magnetic dilution with approximately invariant electrical interactions in the solutions. These experiments were performed with a 2 mole/liter concentration of $MnCl_2 \cdot 4H_2O$ and the following percentages of water by weight in the solvent: 0.6, 5.5, 10.6, 20.6, 40.6, 55.6, 70.6, 85.6 and 100%. The viscosity measured by a Höppler viscometer at 20° C varied monotonically from 3950 centipoise for a glycerin solution to 1.95 centipoise for an aqueous solution. When the viscosity is reduced by a factor of 2000 the relaxation time decreases by a factor of only about 3. For example, when H = 3600, ρ_L decreases from 4.4×10^{-8} to 1.4×10^{-8} sec. b/c, which for a solution of $MnCl_2 \cdot 4H_2O$ in water equals 1.8×10^6 oe^2, increases to 5×10^6 oe^2 as glycerin is added. b/c = 5×10^6 oe^2 corresponds to the presence of equal numbers of moles of glycerin and $MnCl_2 \cdot 4H_2O$ in the solution. With further increase of the glycerin content b/c decreases to 2.5×10^6 oe^2 (for a pure glycerin solution). This dependence of b/c on the composition of the solvent can be attributed to the presence of mixed water-glycerin solvate envelopes around manganese ions which decrease the electric field symmetry at the ions.

To study the dependence of ρ_L on magnetic dilution without changing the electrical conditions, experiments were performed on a solution of $MnCl_2 \cdot 4H_2O$ at a concentration of 0.45 mole/liter with the addition of 1.35 mole/liter of $CaCl_2 \cdot 4H_2O$. The relaxation times ρ_L are in good agreement with the Brons-Van Vleck formula for $\rho_0 = 1.4 \times 10^{-8}$ sec, p = 0.236 and b/c = 2.35 oe^2.

A more detailed description of the experiments on $MnCl_2 \cdot 4H_2O$ in water-glycerin solutions, and also when a diamagnetic salt is added, will be published elsewhere together with a discussion of the results.

In conclusion I take this opportunity to thank B. M. Kozyrev for directing and assisting with this work, and V. I. Avvakumov for participating in a discussion of the results.

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THE ELECTRICAL CONDUCTIVITY OF GERMANIUM IN HIGH ELECTRIC FIELDS AT LOW TEMPERATURES

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The electrical properties of single-crystal germanium doped with Sb, Bi, and Zn were studied in the temperature range 2 — 10° K. In high electric fields the electrical conductivity of germanium displays three regions, in the last of which (the so-called "breakdown" region) a sharp conductivity increase is observed. The phenomenon of breakdown is associated with the development of an avalanche in the conduction band, independent of the conduction mechanism in the germanium at low temperatures. The product of E_b and the mobility μ is a function of I/kT, where I is the ionization energy of the impurity in the germanium. The effect of various factors (temperature, magnetic field) on E_b is mainly due to their influence on the carrier mobility. The dependences of E_b and resistance on magnetic field are found to be similar in character.

THE aim of the present work was to study the phenomenon of the sharp conductivity increase in germanium at low temperatures in fields of the order of several volts per centimeter.

Originally this phenomenon was discovered at hydrogen temperatures,[1] but it was interpreted incorrectly. In 1953, Sclar, Burstein et al.[2] discovered the phenomenon at helium temperatures, so sharply displayed that they defined it as "breakdown" and attributed it to the effect of the impact ionization of neutral impurity atoms by electrons. In 1955 Darnell and Friedberg established the dependence of E_b on magnetic field.[3]

It was of interest to study more carefully the mechanism of this phenomenon, which is analogous to breakdown in dielectrics, but which differs in that it takes place reversibly. This interest is strengthened by the fact that the conductivity behavior in high electric fields can apparently throw light on the conduction mechanism in germanium at low temperatures in a region where an anomaly exists in the dependence on the resistance on temperature.[4]

After the present work had begun, the work of Sclar and Burstein[5] and Koenig and Gunther-Mohr[6] became known to us; the problems posed and their method of solution closely resembled those of our investigations.

EXPERIMENTAL PART

Several series of germanium specimens (of different origin) doped with antimony and bismuth were studied, as well as two specimens cut from one section of a bar doped with zinc (for details of the specimens, see Tables I and II).

A series consisted of crystals cut from one bar in a plane perpendicular to the direction of crystal growth [111]. The specimens were ground, etched in boiling hydrogen peroxide and repeatedly washed in triply-distilled boiling water. The specimen length usually exceeded the width by a factor of not less than four. Contacts were made using spectrographic-purity gold and were soldered to the faces of the specimen by a technique of prolonged heating (an hour at 300° C) in a high vacuum. The massive current contacts completely covered the ends of the specimen, but the potential contacts for Hall effect measurements consisted of gold beads with areas of contact 0.2 — 0.05 mm^2 at a distance of 2 — 3 mm. from one another. After soldering the contacts to the specimen, it was etched and washed again.

The specimen prepared for the experiment was mounted in an apparatus in which it was possible to make prolonged measurements at constant temperatures in the range 1.5 — 77° K. The glass bulb 1 (Fig. 1) was immersed in liquid helium or hydrogen and was evacuated before the experiment to a pressure of ~0.01 mm. Hg. Thermal contact between the specimen and the helium bath was made with the help of the copper "cold" conductor 2. To increase the specimen temperature relative to the external bath, a wire heater 3 was used, wound on the small cylinder of fused quartz 4 which served

TABLE I. Details of specimens

Series	Specimen	Doping impurity	Impurity concentration	ρ^*, ohm/cm	E_b^*, v/cm
Bi-I	Bi-1	Bi	$7.6 \cdot 10^{14}$	$\sim 10^8$	5
	Bi-2	Bi	$1 \cdot 10^{15}$	10^8	5
	Bi-3	Bi	$2.0 \cdot 10^{15}$	$1.1 \cdot 10^9$	10.2
	Bi-4	Bi	$5.0 \cdot 10^{15}$	$1 \cdot 10^7$	20
Sb-I	Sb-1	Sb	$7.6 \cdot 10^{14}$	$1.2 \cdot 10^8$	6.4
	Sb-2	Sb	$1 \cdot 10^{15}$	$1.3 \cdot 10^8$	7.5
Sb-II	Sb-B	Sb	$7.5 \cdot 10^{13}$	$5 \cdot 10^9$	14—16
	Sb-C	Sb	$1.6 \cdot 10^{15}$	$1 \cdot 10^{11}$	38—41
	Sb-H	Sb	$2.75 \cdot 10^{15}$	$9 \cdot 10^8$	56—70
	Ge-B	Not doped	$\sim 10^{13}$	$1.2 \cdot 10^9$	13
	U-20	" "	$\sim 10^{15}$	$8 \cdot 10^8$	9

*At 4.2°K.

TABLE II

Specimen	T° K	ρ ohm/cm	R, coulomb/cm³	$R/\rho \sim A\mu$	E_p, v/cm	μ (theoretical)
Bi-1	4.8	$1.6 \cdot 10^7$	$8 \cdot 10^{12}$	$5 \cdot 10^5$	5	$2 \cdot 10^5$
	5.25	$3.2 \cdot 10^6$	$1.6 \cdot 10^{12}$	$5 \cdot 10^5$	5	$2 \cdot 10^5$
Sb-B	4.2	$5 \cdot 10^9$	$1 \cdot 10^{15}$	$2 \cdot 10^5$	14—16	$2 \cdot 10^5$
	5.75	$1.67 \cdot 10^6$	$2.7 \cdot 10^{11}$	$1.6 \cdot 10^5$	14.5	$2 \cdot 10^5$
Sb-H	6.5	$5.1 \cdot 10^6$	$9 \cdot 10^{10}$	$1.76 \cdot 10^4$	35.5	$5 \cdot 10^4$
	7	$1.38 \cdot 10^6$	$3 \cdot 10^{10}$	$2.17 \cdot 10^4$	34	$5 \cdot 10^4$
	7.7	$3.66 \cdot 10^5$	$1 \cdot 10^{10}$	$2.73 \cdot 10^4$	—	$5 \cdot 10^4$
	8.7	$1.14 \cdot 10^5$	$2.4 \cdot 10^{10}$	$2.1 \cdot 10^4$	—	$5 \cdot 10^4$
Zn-1*	11.6	$8.3 \cdot 10^8$	$4.5 \cdot 10^{12}$	$5.5 \cdot 10^4$	31.5	$5 \cdot 10^4$
Zn-2*	13.96	$1.6 \cdot 10^6$	$1.35 \cdot 10^{11}$	$8.5 \cdot 10^4$	22.5	$5 \cdot 10^4$

*The impurity concentration in the zinc-doped specimens was $\sim 3 \cdot 10^{15}$.

as a thermal resistance (equilibrium between the specimen and the heater was established in a fraction of a second).

The specimen was soldered at one end to the cylinder 4 and at the other end through a plate of single-crystal quartz 5 to the calibrated carbon thermomenter 6. The soldered joints to the quartz were made using a silver paste which was deposited on the quartz and baked at a temperature of 500 — 700° K. The use of crystalline quartz ensured good electrical insulation of the specimen from the body of the apparatus, and, simultaneously, good thermal contact at helium temperatures with the thermometer.

Manganin wires (suspended in the vacuum "Staybrite" tube 12 through the polystyrene washers 10) were used for the electrical leads (potential and current) to the specimen.* To reduce thermal conduction along the leads, the lowest insulating washer, consisting of a quartz plate with tungsten seal-ins, was in thermal contact with the external bath. The leads to the heater and thermometer consisted of insulated copper wires immersed in the external bath 13.

For work in the temperature range obtained by pumping the vapors of liquid helium or hydrogen, the apparatus was not sealed off and the specimen was directly in contact with the bath.

Because germanium is very sensitive at low temperatures to infra-red radiation even of long wavelength, the entire apparatus was shielded by a metallic screen 14 maintained at the temperature of the external bath.

The circuit for measuring electrical resistance is shown in Fig. 2. From the potentiometers P_1 and P_2 for coarse and fine adjustment, the voltage from the battery B is applied to the germanium specimen and the calibrated resistance R_N, the value of which could be changed from 10^{11} to 10^2 ohm. The potential drops across R_N and the potential contacts of the specimen were measured using an SG-1M electrometer with sensitivity 2×10^{-2} v. The entire electrometer circuit was mounted on amber and polystyrene insulators and was carefully screened electrostatically.

To measure the Hall voltages and the resistances in a transverse magnetic field, the apparatus

*A similar method of soldering contacts and assembling the apparatus has been used simultaneously in this laboratory by I. A. Kurovaya.

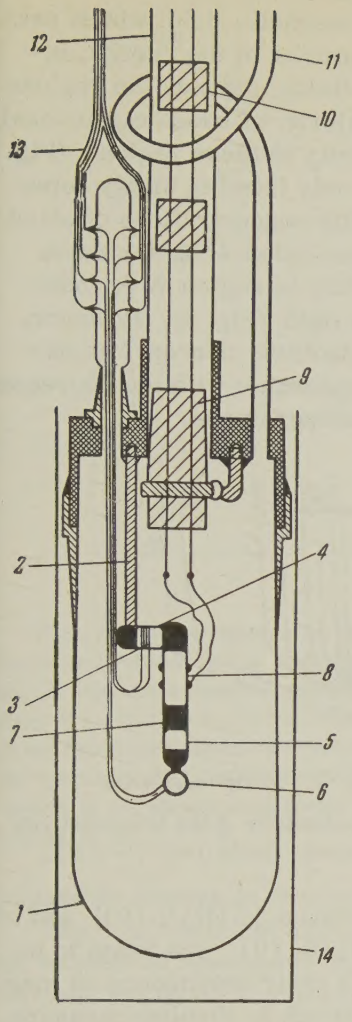

FIG. 1. General view of the apparatus: 1 – glass bulb, 2 – heat conductor, 3 – heater, 4 – fused quartz, 5 – single-crystal quartz, 6 – thermometer, 7 – contacts, 8 – specimen, 9 – quartz washer, 10 – polystyrene washer, 11 – pumping tube, 12 – tube, 13 – leads for thermometer and heater, 14 – metallic screen.

was placed between the poles of an electromagnet which allowed fields up to 10,000 oe to be obtained. The measurements were made using the circuit shown in Fig. 2, the electrometer being usually fed with the voltage from the contacts 1 and 3 (Fig. 2).

To study the dependence of the electrical conductivity on illumination, a miniature incandescent lamp was placed inside the apparatus, the light from which passed through Schott filters (Nos. 2, 4, 6, 7) and illuminated the central portion of the specimen with a sharp beam.

RESULTS

At low temperatures the electrical conductivity of germanium in strong electric fields is characterized by three regions:

A) A region in which Ohm's Law is obeyed;

B) A region of monotonic conductivity increase;

C) A region of sharp conductivity increase or "breakdown."

We took as the "breakdown" field (E_b) the value corresponding to the boundary between

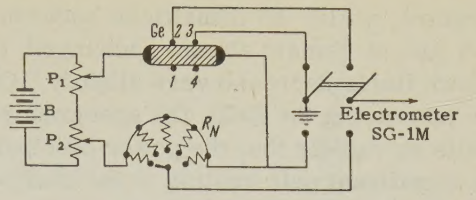

FIG. 2. Electrical measuring circuit: B – battery, P_1, P_2 – potentiometers, R_N – calibrated resistance.

regions B and C (see below, Figs. 3 and 5).

The extent of region A (σ = const.) depends on the specimen purity. For high-resistivity specimens it is significantly smaller than for low-resistivity specimens. Thus, for example, for specimen Sb-B (see Table I) the region A extends to 10% of the breakdown field, for Sb-C to 25% and for Sb-H to 80%.

In region B the field dependence of conductivity is well described for most specimens by the exponential relationship $\sigma \approx \sigma_0 e^{\alpha E}$, where $\alpha = 0.1 - 0.3$, α being larger in high-resistivity than in low-resistivity specimens.* In region C also, the field dependence of electrical conductivity can be represented by an exponential, the slope of which (at low temperatures) is tens of times greater than the slope in region B. On increasing the temperature (Fig. 3) this slope diminishes

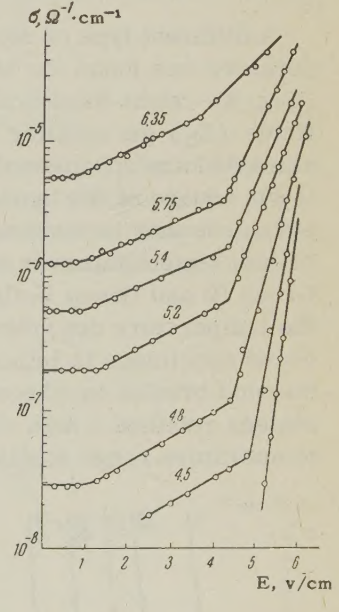

FIG. 3. The electrical conductivity of specimen Bi-1 in high electric fields at various temperatures (°K).

and the change-over between regions B and C is found to become less sharp. (For germanium doped with antimony and bismuth, the breakdown fields E_b already became difficult to determine at temperatures near 7° K). On increasing the

*Departures from the relationship $\sigma \approx e^{\alpha E}$, were found only in the zinc-doped specimens, the conductivity of which was approximately proportional to E^2 in region B.

temperature, whilst the transitions between regions A and B remain almost unchanged, the breakdown fields decrease very slightly. On further increasing the field, the specimen resistance falls so rapidly that the power dissipated causes significant self-heating — the start of "thermal" breakdown.

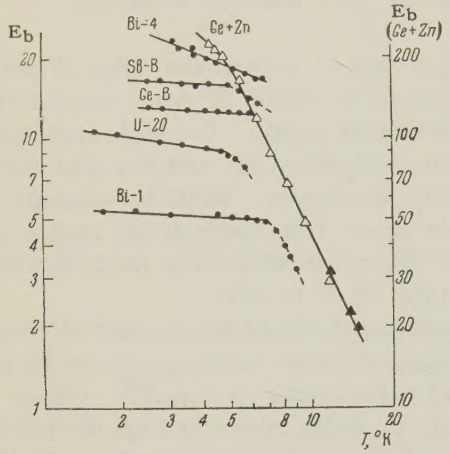

FIG. 4. Dependence of E_b on temperature. The left-hand ordinate scale refers to Sb- and Bi-doped specimens, and the right-hand scale to Zn-doped specimens. ▲ – Results for Zn-1, Zn-2 obtained using circuit in Fig. 2. Δ – Results for Zn-1, Zn-2 obtained not using the potentiometric method of measurement; the broken lines show regions where the specimens might possibly be overheated.

A different type of dependence of E_b on temperature was found for zinc-doped specimens (Fig. 4 — right-hand ordinate axis). The critical fields (E_b) get smaller with increase of temperature and are approximately inversely proportional to the square of the temperature; at hydrogen temperatures they become as small as those found at helium temperatures in specimens doped with Group III and Group V elements. The rapidity of the temperature dependence $E_b(T)$ for zinc-doped specimens is impossible to attribute to a thermal breakdown phenomenon, because the specimens retained a high resistance in the whole temperature range studied.

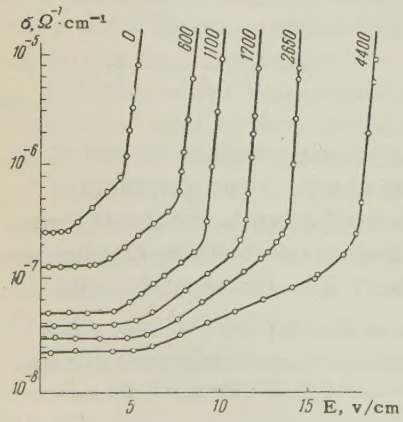

FIG. 5. The electrical conductivity of the high-resistivity specimen Bi-1 in various magnetic fields (Oe); T = 5.25°K.

The application of a magnetic field, whilst preserving the general character of the curves, increases the breakdown fields, extends the regions A and B, and in the majority of cases continuously diminishes the conductivity in these regions (Fig. 5). A different behavior is only found in highly doped specimens (with impurity concentrations of about 5×10^{15}) at temperatures below 4° K; for these specimens the conductivity in region A is independent of the magnetic field (Fig. 6). However, in this case also, a conductivity decrease is observed in region B, comparable with the decrease in region A at higher temperatures.

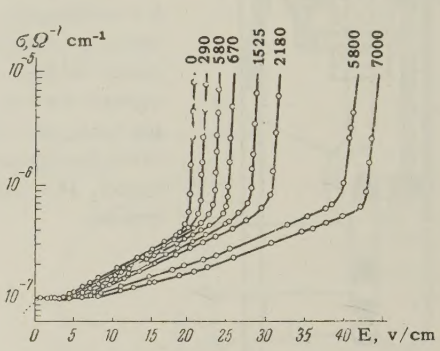

FIG. 6. The electrical conductivity of the low-resistivity specimen Bi-4 in various magnetic fields (Oe); T = 4.2°K.

The changes of resistance $[\rho(H)/\rho(0)]$ and of breakdown field $[E_b(H)/E_b(0)]$ are found to be similar in the manner of their dependence on magnetic field (Fig. 7), although in absolute measure the increase of $\rho(H)/\rho(0)$ can exceed the increase of $E_b(H)/E_b(0)$* by a large factor (especially in high-resistivity specimens).

The Hall voltage measurements in high electric fields (Fig. 8) show that the Hall constant R barely changes during the transition into region B. Change of Hall angle within the accuracy of the experiment only takes place in the immediate vicinity of E_b.

On illuminating the specimen, the j-E characteristics are displaced upwards along the ordinate axis and are distinct from the curves obtained without illumination. E_b remains approximately unchanged.

The curves described were reversible and reproducible in a series of experiments under identical conditions. Resoldering the contacts, re-etching the surface and re-assembling the apparatus produced discrepancies smaller than 10% (in absolute terms); these were apparently

*It should be noted that the dependence of $E_b(H)/E_b(0)$ on magnetic field H shown in Fig. 7 by circles is observed only using the four-electrode circuit. The use of the two-electrode method gives the dependence shown in Fig. 7 by crosses.

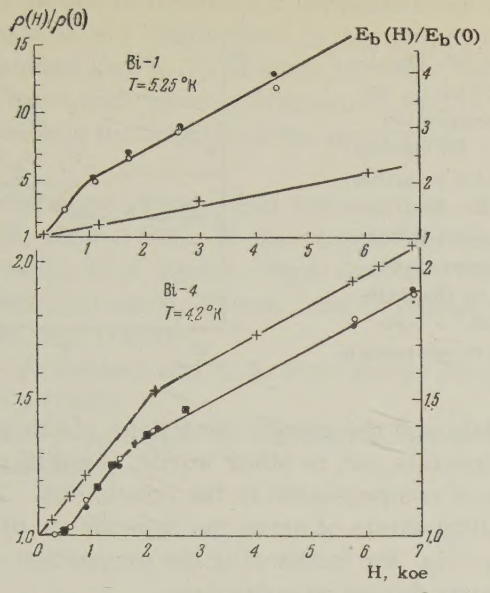

FIG. 7. The dependence of breakdown field and resistance on magnetic field H for specimens Bi-1 and Bi-4. ● – $E_b(H)/E_b(0)$ from measurements obtained using the circuit shown in Fig. 2; + – $E_b(H)/E_b(0)$ from measurements made when not using the potentiometric method; ○ – $\rho(H)/\rho(0)$ in region A for specimen Bi-1, in region B for specimen Bi-4; ■ – increase of resistance in magnetic field for specimen Bi-4 in region A at T = 7°K.

caused by errors in determining the dimensions.*

The bulk nature of breakdown was confirmed by studies on specimens of complex shapes with various ratios of surface to volume along their lengths.

DISCUSSION OF RESULTS

The impact ionization[2,5] of neutral impurity atoms by electrons which have acquired the necessary energy in fields of the order of several volts per centimeter is, apparently, the only possible mechanism of breakdown, because for germanium doped with Group III and Group V elements the Zener mechanism,[7] involving the extraction of electrons into the conduction bands (the tunnel effect), ought only to appear in fields greater than 500 v/cm.

There being no quantitative theory of avalanche development, the results are compared with conclusions obtained from a qualitative consideration of the equilibrium between the rate of energy accumulation by electrons in the field and the rate of

*It should be remarked that in specimens of series SbII a phenomenon of "delayed breakdown" was observed; this meant that for breakdown it was necessary to apply to the specimen a somewhat higher voltage than that for which "breakdown" started to develop. These phenomena could not be attributed to the effect of the contacts.

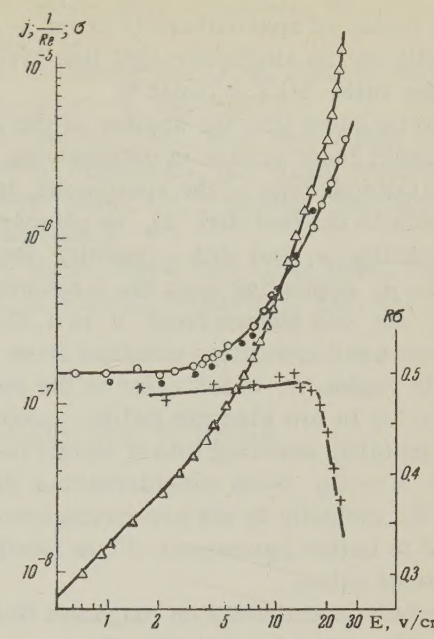

FIG. 8. The behavior of specimen Sb-B in high electric fields (T = 5.75°K). Δ – current density j(E), ● – electrical conductivity σ(E) (in reduced units, ○ – 1/RE, + – Rσ in relative units.

energy dissipation into thermal oscillations of the lattice;[5,8] this leads to the formula

$$E_b \approx \frac{u}{\mu}\sqrt{\frac{2I}{kT}} \qquad (1)$$

or

$$E_b\mu \approx \text{const}, \qquad (1a)$$

where u is the speed of sound, I the ionization energy of the impurity and μ the mobility. The relationship between E_b and μ obtained experimentally (Fig. 9) does not contradict Eq. (1a).

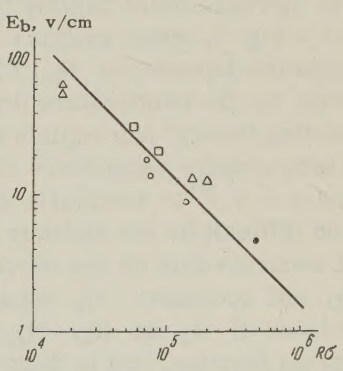

FIG. 9. Dependence of E_b on the mobility Rσ for Δ – the series Sb-II, ● – the specimen Bi-1, □ – the specimens doped with zinc, ○ – results of reference 6; the continuous line is the theoretical curve (from reference 5).

From Fig. 9 it is clear that the E_b values for specimens of germanium doped with antimony, bismuth and zinc — the ionization energies of which

differ by a factor of approximately three — lie satisfactorily on the single straight line given by Eq. (1) if the ratio $I/kT \approx$ const.*

It should be noted that the scatter of the points in Fig. 9, apart from errors in determining the geometrical dimensions of the specimens, is perhaps also due to the fact that E_b is compared not with the mobility μ, but with a quantity $R\sigma$, proportional to μ; depending upon the mechanism of scattering, $R\sigma$ can change from μ to 1.93μ. Also we have used mobilities obtained from measurements in region A, which refer to the specimens' behavior in low electric fields. According to (1), the mobility entering into it should be taken as that for $E \to E_b$. Such considerations, by diminishing the mobility in the pre-breakdown fields, should lead to better agreement of our results with the theoretical curve.

The similar dependences on magnetic field of resistance and critical field (Fig. 7)

$$\rho(H)/\rho(0) \approx kE_b(H)/E_b(0), \qquad k = 1 \div 4, \qquad (2)$$

can also be explained qualitatively by the dependence of E_b on μ, if one takes into account that in region A the resistance increase in a magnetic field is caused only by a fall of mobility:

$$\rho(H)/\rho(0) \approx \mu(0)/\mu(H).$$

That the coefficient k differs from unity is due, it would appear, to the different carrier concentration increases in strong electric fields (in region B) with and without magnetic field. In so far as the carrier concentration increase depends on the energy gathered by the electron in the field E, then the concentration increase in a magnetic field can be slower and can cause $\rho(H)/\rho(0)$ to increase more rapidly than $E_b(H)/E_b(0)$ (Fig. 7, upper graphs).

The temperature dependence $E_b(T)$ is determined, it seems, by the temperature dependence of μ. The existing theory[9] can explain both a weak and a strong temperature dependence of $\mu(T)$ ($\mu \sim T^{1/2}$ and $\mu \sim T^{3/2}$). Comparison with the theory is made difficult by the absence for our specimens of accurate data on the concentration of donors N_D and acceptors N_A separately.

The dependence of E_b on $N_D - N_A$ (Fig. 10) is not a universal function, and is described by a series of curves similar to the curve obtained by Burstein and Sclar.[5] Their character cannot be ascribed to the dependence of the mobility on $N_D - N_A$ and is probably determined by the raw materials and the growth conditions of the germanium crystals, or, in other words, the different degree of compensation in the specimens. The possibility exists of using the dependence of E_b on $N_D - N_A$ for monitoring the properties of specimens during manufacture.

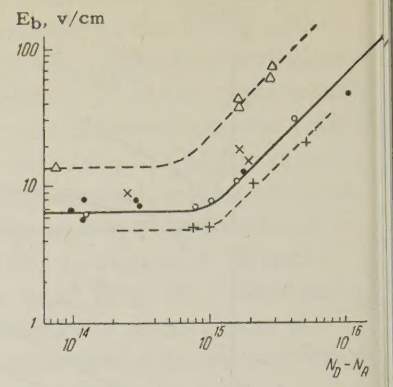

FIG. 10. The dependence of E_b on the concentration $N_D - N_A$ for various specimens: specimen series, o – Sb-1, Δ – Sb-II, + – Bi-1; continuous curve – according to Burstein and Sclar.[5] × – according to reference 6.

The exponential growth of the number of carriers with increase of field in region B, $\sigma \sim \sigma_0 e^{\alpha E}$, where $\alpha = 0.1 - 0.3$, is not explicable in terms of the specimen warming up, because its temperature is maintained constant with an accuracy of 0.01° K, even for the very large current densities. Possibly this increase in the number of carriers is due to their decreased recombination[6,10] in high electric fields.

Apart from breakdown phenomena, the study of high-field conductivity has an interest of its own. At very low temperatures (below 4° K) the conductivity of germanium is determined, as is known, not only by the straightforward effect of the conduction band, but also by certain other processes, in particular those connected in a complicated fashion with the impurity concentration.

The conductivity of specimens with large impurity concentrations (greater than 5×10^{15}) is due to conduction in the sharp impurity zone formed by the crossing of the impurity center wave functions.

The fact that the resistance is observed experimentally to be independent of magnetic field (in region A) for highly-doped specimens (Fig. 6), supports the idea of a sharp impurity zone with small carrier mobility. The resistance increase found in a magnetic field (in region B) (Fig. 6) can be attributed to the increased role of the conduction band in high electric fields shunting the impurity conduction mechanism. The latter circumstance enables one to deduce that, whatever the conduction mechanism of germanium in low electric fields, the phenomenon of breakdown is connected with avalanche development in the conduction band.

*The ionization energies of the impurities used were calculated from the temperature dependence of specimen resistance.

The author in conclusion expresses his deep gratitude to A. I. Shal'nikov, B. M. Vul, S. G. Kalashnikov and L. V. Keldysh for their interest in the work and valuable discussion, and to V. G. Alekseevaya for supplying the specimens.

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SMALL EFFECTS OF SOLAR FLARES AND THE ENERGY SPECTRUM OF PRIMARY VARIATION OF COSMIC RAYS

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The relation between the intensity of cosmic-ray neutrons and chromospheric flares on the sun was studied from the data obtained at four stations located at various latitudes. By the epoch-superposition method, a small flare effect has been found at each of these stations. The energy spectrum of particles of the supplementary flux has been determined with the aid of the coupling constants.

As is well known, the appearance of sun flares is accompanied by a small increase in the intensity of cosmic rays.

The author carried out investigations in order to 1) explain the character of variation of the neutron component intensity in cosmic rays during small solar flares and 2) determine the energy spectrum of the additional flux of cosmic rays. Data obtained at the stations listed in the table were utilized.

The author participated in collecting the data at the first station. Results of all measurements were corrected for the barometric effect.

Since variations of the intensity of cosmic radiation for each case are small and their magnitude is not bigger than that of the statistical fluctuations, the data were averaged by the epoch superposition method.[1] Two-hour intervals, including the time of the beginning of the flares, were taken as zero time. Readings of all corresponding two-hour intervals before and after this period were then added. In such a manner, data for 41 flares with an index of 2 and higher,* observed from July to November 1957 between 22 h 00 min and 14 h 00 min World Time, were reduced.

Station	Altitude, m	Geographical coordinates		Geomagnetic coordinates	
		Latitude (N)	Longitude (E)	Latitude	Longitude
Alma-Ata (U.S.S.R.)	806	43°14′	76°51′	33°	150,5°
Rome (Italy)	60	41°48′	12°36′	42.4°	92,1°
Göttingen (Western Germany)	2960	47°8′	09°5′	49,1°	93.4°
Göttingen (Western Germany)	273	51°32′	09°56′	52.3°	93,6°

Results are given in Fig. 1. The following facts can be noted:

1) A decrease in the intensity of 0.1 — 0.2% is observed before the flares.

2) The flare effect $\delta I/I$ (where I is the intensity of the neutron component of cosmic rays for the given station) is greater at high latitudes:

Geomagnetic latitude (N)	33.0°	42.4°	49.1°	52.3°
$\delta I/I$, % of mean	0.50±0.08	0.60±0.08	1.60±0.08	1.00±0.06

3) The neutron-intensity maximum occurs after the solar flare with a delay (2 — 5 hours) which increases with the latitude;

4) The duration of intensity decrease is larger at high latitudes: thus, at the latitude of 33° (N), the duration of decrease is 5 — 6 hours, while at the latitude of 52.3° (N) it is ~ 10 hours.

5) The duration of the intensity decrease is larger by 20 — 30% than the rise time.

Several of these effects were observed for large flares,[1,2] and are mentioned by Dorman[1] and by Sekido and Murakami.[2] Since the effect of single flares is not larger than the statistical fluctuations of the intensity for the given time interval, the data of the Alma-Ata station of period equal to the total time of the investigated 41 flares were reduced as a check. These data were selected from days when flares were absent, and were reduced in the same way as for flares. Results of the analysis are presented in Fig. 2. It can be seen from the figure that, within the limits

*The index characterizes the intensity of flares according to the scale: 1, 1^+, 2, 2^+, 3, 3^+.

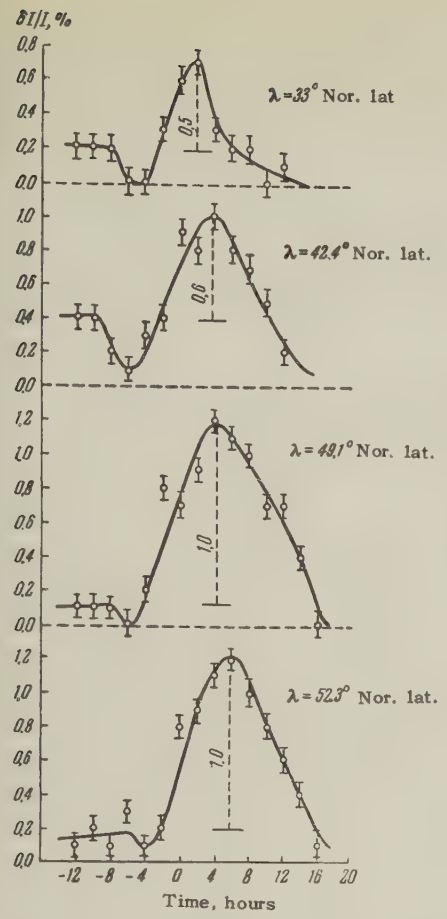

FIG. 1. Variation of the intensity of cosmic-ray neutrons, connected with solar flares.

of statistical errors, the effect has not been observed.

As has been shown by Dorman,[1] one can determine the spectrum of the primary cosmic radiation in the energy range subjected to the influence of the magnetic field of the earth from measurements of any single component at various latitudes by the method of the coupling constants.

The determination of the energy spectrum of primary variation of cosmic rays was carried out according to the formula (see reference 1):

$$\frac{\delta D}{D} = [\delta N^i_{\lambda_k}(h_0) - \delta N^i_{\lambda_{k+1}}(h_0)] \,/\, N^i_{\lambda_0}(h_0) \int_{\varepsilon^{min}_{\lambda_k}}^{\varepsilon^{min}_{\lambda_{k+1}}} W^i_{\lambda_0}(\varepsilon,\ h_0)\, d\varepsilon,$$

where $\delta D/D$ is the ratio of the differential energy spectrum of the additional radiation to the spectrum of the unperturbed primary cosmic radiation, $\delta N^i_{\lambda_k}(h_0)$ is the variation of the intensity of the i-th component at the latitude λ_k and altitude h_0, $\epsilon^{min}_{\lambda_k}$ is the effective value of the minimum energy resolved by the magnetic field of the earth (geomagnetic cutoff), and $W^i_{\lambda_0}(\epsilon, h_0)$ is the coupling constant between the primary and the observed variations of the i-th component at latitude λ_0.

The spectrum was calculated for various moments of time. The results of calculations at two hours after the beginning of flares, and also according to the maxima of the effect, are given in Fig. 3. Graphs of the dependence of $\delta D/D$ on the energy have, within the limits of errors, the same slope. This indicates that there was no fast redistribution of particles energy in the additional flux during that time. Extrapolation of the straight lines to the x axis determines approximately, the upper limit of the energy spectrum of primary variation, which can be seen in Fig. 3 and lies in the vicinity of 14 — 17 Bev.

With respect to the lower limit of the energy spectrum, one can state the following (provided that the value of the effect at 49.1° (N) latitude is not partly due to the altitude of the station, situated at an altitude of 2960 m above sea level): within the limits of errors, the effect at latitudes of 49.1 and 52.3° (N) is identical; hence particles with total energy not lower than ~ 3 Bev (geomagnetic cut-off at 49.1° (N) latitude were present in the additional flux.

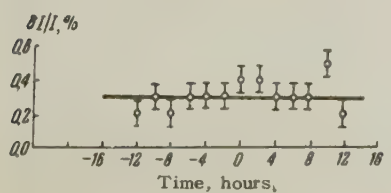

FIG. 2. Variation of neutron intensity in days without solar flares.

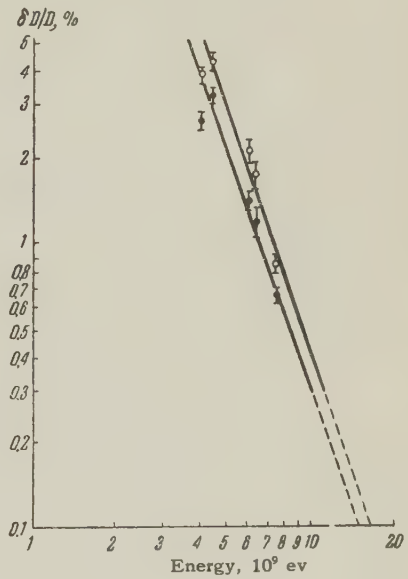

FIG. 3. Spectrum of primary variations, obtained from the effect of small solar flares on the neutron component. The y axis represents the ratio of the energy spectrum of the additional radiation to that of the undisturbed primary cosmic radiation: ● — from the data obtained 2 hours after the onset of flare, ○ — from the data at the maximum.

The observed spectrum is in agreement with the spectrum calculated for the large flare of cosmic rays which occurred on the 23rd of February 1956.[1] This agreement indicates that both the small and large increases of the intensity of the neutron component of cosmic rays are probably due to the additional flux of particles of the same energy and the same nature.

In conclusion, the author would like to express his deep gratitude to L. I. Dorman for helpful advice, to the students K. Tarasova and V. Pivneva who took part in the work, and also to the coworkers of the stations in Rome and Göttingen, whose results of measurements, submitted to the World Center for Collection of Data of the International Geophysical Year, were used as the basis for the present work.

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ON THE EXISTENCE OF THE NEGATIVE NITROGEN ION

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A beam formed by the passage of positive nitrogen ions through a gaseous target has been found to contain small amounts of negative nitrogen ions. The cross section for the $N^+ \rightarrow N^-$ process is estimated as 1.9×10^{-22} cm^2. No negative molecular nitrogen ions are observed.

THE question of how the atoms in the periodic system form stable negative ions has been considered at great length in a number of papers by Dukel'skiĭ and his co-workers.[1-5] The experimental data seem to point to the following rule: atoms with unfilled electron shells form stable negative ions while atoms with filled shells do not form negative ions. Only one exception to this rule has been observed at the present time: the nitrogen atom, which has a $2p^3$ configuration in its outer electron shell. It has been shown by a number of authors[3,6,7] that N^- ions are not produced in a gaseous discharge in nitrogen. Dukel'skiĭ and Zandberg,[3] in observing the spectrum of negative ions from a discharge in NH_3, observed a weak line at mass 14; however, as indicated by the authors, a line with an apparent mass of 14 may be produced by the NH^- ions which result from the dissociation of NH_2^- ions in the space between the ion source and the magnetic mass analyzer. It should be noted, however, that the absence of N^- ions in the plasma of a gaseous discharge is still not proof of the nonexistence of this ion; because of the low electron affinity of the nitrogen atom the density of negative nitrogen ions in a plasma may be very small.

We have made an attempt to observe N^- ions resulting from the $N^+ \rightarrow N^-$ process in the passage of a beam of N^+ ions through matter. As we have already indicated,[8-10] the effective cross section for the $I^+ \rightarrow I^-$ process in single collisions of a great many positive ions with heavy inert gas atoms reaches the gas-kinetic cross sections and, in those cases in which a comparison is possible, turns out to be considerably greater than the effective cross section for the formation of negative ions in collisions of electrons with gas molecules. Thus, the former would appear to be the most effective method of observing negative ions. The validity of this statement has been verified by the work of Dukel'skiĭ et al,[11] who used this method to observe metastable He^- ions, which have a very small electron affinity (0.075 ev[12]).

In the search for N^- use was made of the double mass spectrometer which has been described earlier.[13] A high-frequency ion source was used for obtaining N^+ ions. The beam of N^+ ions, with an energy of 34 kev, was analyzed by a magnetic mass monochromator and directed into the collision chamber, which was filled with krypton. In the mass spectrum of the beam, in addition to the line at mass 14, we observed a series of lines due to contamination of the nitrogen by gases evolved from the walls of the source chamber and by gaseous dissociation products of the diffusion pump oil. In particular, close to the line at mass 14 the following lines were observed: 12 (C_{12}^+), 13 ($C_{13}^+ + C_{12}H^+$), 15 ($N_{15}^+ + N_{14}H^+$), 16 ($O_{16}^+ + C_{12}H_4^+ + N_{14}H_2^+$), 17 ($O_{16}H^+ + N_{14}H_3^+$), and 18 ($O_{16}H_2^+$).

The resolving power of the mass monochromator was sufficient for complete separation of the line at mass 14 from the nearby lines at 13 and 15. The beam which passed through the collision chamber was analyzed by means of a magnetic analyzer. The negative ion current was measured with a vacuum-tube voltmeter (sensitivity 10^{-14} amp/div).

The first experiments, at a current strength of the order of 10^{-7} amp, indicated that the beam contained a small amount of negative ions of mass 14; the number of negative ions increased with increasing pressure of the krypton in the collision chamber. The mass-spectrometer method (cf. references 8 and 13) was used to measure the effective cross section for the conversion of a positive ion of mass 14 into a negative ion of the same mass. This cross section was found to be 3.2×10^{-22} cm^2. However, this result, in itself, is not necessarily proof that the effect is due ex-

clusively to the $N^+ \rightarrow N^-$ process. In an ion beam of mass 14, in addition to the N^+ ions there may be CH_2^+ ions or fragment ions with apparent mass 14 which arise as a result of the dissociation of heavier positive ions in the space between the ion source and the mass monochrometer. Using the formula $m^* = m_1^2/M$ (where m^* is the apparent mass of the fragment ion, m_1 is the mass of the fragment ion and M is the mass of the ion before dissociation) we find that the quantity m^* can be near 14 for O^+ ions resulting from $H_2O^+ \rightarrow O^+ + H_2$ ($m^* = 14.2$) and for NH^+ ions resulting from $NH_2^+ \rightarrow NH^+ + H$ ($m^* = 14.1$). Thus the negative ions of mass 14 can result from two-electron charge exchange on CH_2^+ and the fragments O^+ and NH^+ in addition to the $N^+ \rightarrow N^-$ process.*

The effect of O^+ on the size of σ_{1-1}^{14} was established by investigating the effect of the H_2O^+ ion current in the primary beam on the measured value of this cross section. The value 3.2×10^{-22} cm^2 indicated above for this cross section was measured when the H_2O^+ ion current was 2.4×10^{-10} amp. As a result of extended processing of the walls of the source chamber and careful purification of the nitrogen it was possible to reduce the H_2O^+ current to 4.7×10^{-11} amp. At this value of the H_2O^+ current σ_{1-1}^{14} was found to be 1.9×10^{-22} cm^2. A simple calculation indicates that for this value of the H_2O^+ current the admixture of residual O^+ ions has no effect on σ_{1-1}^{14}. There still remains the possibility of contamination of the N^+ beam by CH_2^+ ions and residual NH^+ ions, the existence of which was verified by the presence of N^+ and H_1^+ in the beam; these arise as a result of the following dissociation processes: $CH_2^+ \rightarrow CH + H^+$ and $NH^+ \rightarrow N + H^+$. We have carried out additional experiments in order to determine the cross sections for $CH_2^+ \rightarrow CH_2^-$ and $NH^+ \rightarrow NH^-$ in krypton at positive ion energies of 34 kev; these cross sections were found to be 5.3×10^{-19} cm^2 and 5.3×10^{-18} cm^2 respectively. On this basis an experimental estimate was made of the contamination of the primary beam by CH_2^+ ions.

Calculations carried out on the basis of these measurements indicate that only part of the negative ions observed in the beam can result from the $CH_2^+ \rightarrow CH_2^-$ and $NH^+ \rightarrow NH^-$ processes; hence part of the effect is due to the $N^+ \rightarrow N^-$ process, the cross section of which is estimated as 1.1×10^{-22} cm^2.

The extremely small value for the σ_{1-1} cross section for the $N^+ \rightarrow N^-$ process indicates that the binding energy of the excess electron in the N^- ion must be very small, in agreement with the estimates of this quantity made on the basis of empirical formulas for isoelectronic sequences and the nature of the electronic configuration of this ion.[17]

In addition to the measurements described above an attempt was made to observe negative molecular nitrogen ions; the $N_2^+ \rightarrow N_2^-$ process in krypton was used for this purpose. However, no N_2^- ions were observed; it follows that the cross section for the $N_2^+ \rightarrow N_2^-$ process is smaller than 1.5×10^{-22} cm^2.

*CH_2^- ions have been observed by a number of authors[14-16] and the NH^- has been observed by Dukel'skiĭ and Zandberg.[3]

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PHOTOPRODUCTION OF LOW ENERGY CHARGED PIONS FROM COMPLEX NUCLEI

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The yields of charged photomesons with energies from 0 to 3 Mev at an angle of 90° to a photon beam were investigated for Be, C, Al and Cu nuclei. The maximum photon energy was 265 Mev. The dependence on the negative π-meson yield and the ratio of the positive versus negative π yield on the atomic number were found. Comparison of the experimental data with the physical calculations of Baldin and Lebedev[8] gave the result that the mesons are formed from the nucleons on the nuclear surface.

It is known that the experimentally-observed dependence of the yield of fast charged π mesons on the atomic number is described by the $A^{2/3}$ law. (references 1 — 3). This character of the dependence can be explained by the fact that the creation of mesons comes from only the nucleons on the nuclear surface and that the creation of mesons inside the nucleus is strongly suppressed. However, there exists another point of view, according to which the creation takes place in the whole volume of the nucleus, but the mesons formed by the inner nucleons are absorbed on passing through the nucleus. The truth of the last model can be tested by investigating the yield of mesons whose mean free path in the nuclear matter exceeds the dimensions of the nucleus. The yield of such mesons should be proportional to the number of nucleons involved in their creation. However, recently published experimental results on the photoproduction of 12 Mev positive mesons by various nuclei[4] did not give a unique solution to this problem, even though it was found that the meson yield was proportional to $A^{2/3}$. The authors calculate that such a dependence can be got from the results of the Coulomb field on the yield of mesons created in all the nucleons of the nucleus.

In the present work we investigate the yield of positive and negative photomesons with energies from 0 to 3 Mev for an angle of 90° ± 20° to the direction of the photons in the laboratory system. The work was carried out on the Academy of Science Physics Institute synchrotron with maximum photon energies of 265 Mev.

The gamma-ray beam was collimated by a lead block with a rectangular aperture 3 × 21 mm and was purified of charged particles by a magnetic field of 7000 oersteds. For targets we used thin foils of the following elements: Be — 0.0659 g/cm^2, C — 0.0446 g/cm^2, Al — 0.0377 g/cm^2, and Cu — 0.141 g/cm^2. The average angle for multiple Coulomb scattering in targets of these thicknesses did not exceed 5°; the energy losses were 0.1 Mev for 3-Mev mesons. The targets were strengthened by thin caprone filaments (diameter 0.0015 mm) located outside of the beam.

The mesons were detected by photographic plates, NIKFI type K, with emulsion thickness 400 μ. During irradiation, the target and plates were placed in a vacuum chamber whose front wall lay in a magnetic field. Lead and graphite shields guarded the chamber from background effects.

The main outline of the experimental setup is given in Fig. 1.

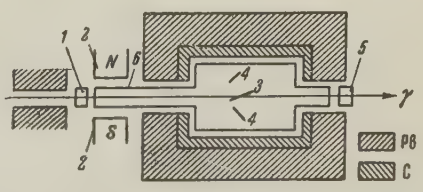

FIG. 1. Schematic diagram of the experimental setup. 1, 5 — ionization chambers, 2 — magnet, 3 — target, 4 — photographic plates, 6 — vacuum chamber.

The plates were examined twice. The effectiveness of the double examination was 96 — 98%. In the scanning, the tracks of π's were chosen which ended in stars (negative π mesons) and in π-μ decays (positive mesons).

The distribution of stars according to the number of prongs corresponded well with the distribution known from other work. Consequently, one can take as rather securely determined the number of negative mesons accompanying visible stars. To determine the entire number of π^- mesons one has to take into account the cases in

which the capture of a meson by an emulsion nucleus is not accompanied by star formation. The probability of observing such cases would be greatly lowered by the strong proton background, but its contribution to the total number of disintegrations depending on π^- mesons is well known and amounts to 27%.[5] The quantity of π^+ mesons was determined by introducing geometrical corrections to the number of observed π-μ decays. The π energy was measured by its residual range in the emulsion. The error in the energy measurement did not exceed 3%.

In calculating the cross section for meson formation on nucleons, no account was taken of charge-exchange and inelastic effects in the scattering of mesons by the nucleons of the nucleus. As shown in references 6 and 7, these effects are rather small for small angles.

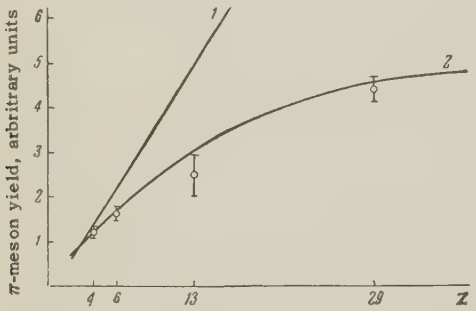

FIG. 2. Dependence of the output of negative mesons on the atomic number. Solid curves are calculated: 1 – for meson creation on all the nucleons in the nucleus, 2 – for nucleons of the surface layer of the nucleus. Statistical errors are shown.

The cross sections for the photoproduction of negative mesons are shown in Fig. 2. For comparison, the theoretical curves for 2-Mev mesons from the work of Baldin and Lebedev[8] are given. Curve 2 corresponds to meson production on nucleons of the outer layer of the nucleus, curve 1 to production throughout the nuclear volume. Both curves were calculated to take into account the interactions of the outgoing mesons with the nuclear and Coulomb fields of the nucleus. As seen on Fig. 2, the experimental points lie well on curve 2, thus confirming the model according to which meson creation inside the nucleus is strongly suppressed. Attention should be called to the absence of maxima for nuclei with excess neutrons, in contrast to previous work[2] in which the dependence of 65 Mev meson output on atomic number was investigated.

Figure 3 shows the ratio of the output of positive and negative mesons as a function of atomic

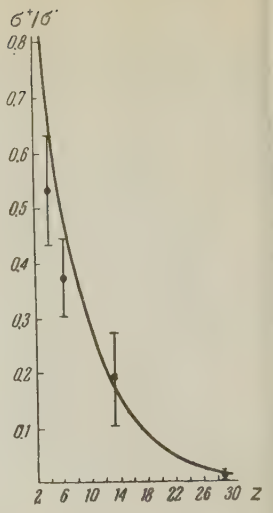

FIG. 3. Dependence of the ratio of outgoing π^+ and π^- mesons on the atomic number. Curve calculated by Baldin and Lebedev.

number. For our energies, the π^+ to π^- ratio decreased with increasing A. This effect can be explained by the interaction of the outgoing mesons with the Coulomb field of the nucleus, which for slow mesons leads to a big difference in the output of mesons of different signs. This interaction was studied by Baldin and Lebedev (private communication); the experimental points agree with their curve.

In ending, we must express our thanks to V. I. Veksler for guiding the work, and also to A. M. Baldin and A. I. Lebedev for reviewing the results.

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AN INVESTIGATION OF THE EXCITED STATES OF Re^{187}

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The excited-state angular momenta and multipole types of the γ transitions of Re^{187} have been determined by measuring the angular correlation of the 552 — 134, 480 — 134, and 72 — 134 kev γ cascades. A value of $(2.2 \pm 0.5) \times 10^{-2}$ was obtained for the ratio of the E2 and M1 amplitudes of the radiation mixture $\delta^2 = I(E2)/I(M1)$, in the γ transition with an energy of 134 kev. The relative intensity of the 552-kev γ line was determined. The internal conversion coefficient for the K-shell γ transition with an energy of 134 kev was measured.

INTRODUCTION

THE decay of W^{187} and the nuclear level scheme of Re^{187} has been repeatedly investigated by various methods.[1-7] The results of these investigations, which characterize the properties of the β transitions of W^{187} and the γ transitions of Re^{187}, are illustrated by the table and by the decay scheme of W^{187} in Fig. 1.

One may assume that the sequence of γ transitions, spins and parities of the ground state and the first excited state, and also the multipole types of the γ_1 and γ_3 transitions of the Re^{187} nucleus have been safely established. The information on the multipole types of the other γ transitions is contradictory.

The present work had the purpose of obtaining additional data relative to the Re^{187} level scheme with the aid of scintillation coincidence methods.

APPARATUS

A block diagram of the measuring setup is shown in Fig. 2. The detection of the γ radiation was made by cylindrical crystals of NaI(Tl) of diameter 30 mm and heights 13 and 20 mm. The crystals were mounted on the photocathodes of two FEU-S photomultipliers. After amplification

γ-transition	Energy, kev	Relative intensity	Internal conversion coefficient	K/L	Multipolarity
γ_2	72.00±0.007 [1]				$E1$ [2]*, $E2$ [7]
			α=2.3 [9]		$M1$ or $M1+E2$ [3]
		0.45 [8]	α_K=2.0±0.2*	5.5±0.5 [3]	$M2$ or $M1+E2$ [4]
γ_1**	134.25±0.018 [1]	0.45 [6]	α_K=2.5 [4]	5.0±0.5 [6]	$M1$ [2,6]
			α_L=0.35 [9]		$M1+E2$ [7]*
γ_6***	206.3 [3]			4.0±0.5 [3]	$M1+E2$ [6]
				4.5±0.5 [6]	
γ_3	479.52±0.19 [1]	1.00 [8]	α_K=0.022 [8]	4.2±0.5 [3]	$E2$ [2,3,6]*
		1.00 [6]		4.0±0.5 [6]	
γ_4	552 [2,6]	0.31 [8]			$E1$ [2]*
		0.40 [6]			
		0.10±0.02*			
γ_7	618.89±0.31 [1]	0.42 [8]		4.0±1.0 [3]	$E2$ [3,6]
		0.30 [6]		3.8±1.0 [6]	
γ_5	686.06±0.38 [1]	1.48 [8]		5.0±1.0 [3]	$E2$ [3], $E1$ [2]*
		1.85 [6]			

*Results of the present work.

**$L_3/L_1 = 1/60$,[3] $L/M = 3.5 \pm 0.5$.[6] The ratio of the radiation amplitudes of the mixture E2 and M1 in the 134-kev γ transition amounts to $(2.2 \pm 0.5) \times 10^{-2}$.

***The ratio of the intensity of the 206-kev γ-quanta to the intensity of the unconverted 134-kev γ-quanta is less than $1/30$.[2] The lifetime of the 206-kev level energy is $T_{1/2} = 0.51 \pm 0.02$ μsec.[10]

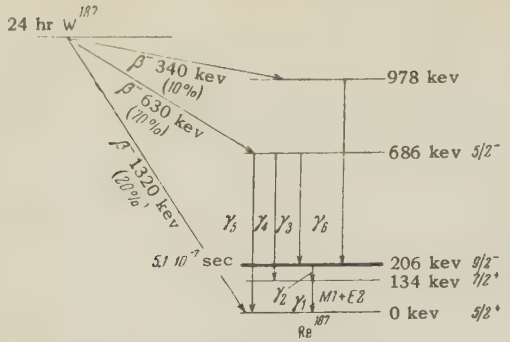

FIG. 1. Decay scheme of W^{187}.

and amplitude selection with the help of an amplitude analyzer, the pulses from the light flashes in the crystals entered a multi-channel delayed-coincidence circuit. The circuit made possible measurements with a delay time of from 0 to 1 μsec. The number of channels of the coincidence circuit used varied from 3 to 6, depending on the nature of the problem. The resolving power τ of each channel also varied from 1.5×10^{-8} to 10^{-7} sec. A single count was registered by scalers.

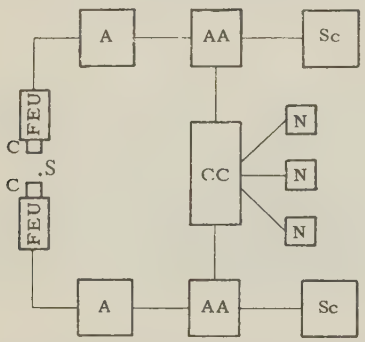

FIG. 2. Block diagram of the measuring set-up: S – source; C – crystals of NaI(Tl); FEU – photomultipliers; A – amplifiers; AA – amplitude analyzers; CC – multichannel delayed coincidence circuit; Sc – scaler; N – electromechanical counters.

The apparatus allowed the taking of instantaneous and delayed coincidence spectra, the determination of internal conversion coefficients and the measurement of angular correlations. The lifetime of the metastable state of Re^{187} was measured earlier in similar apparatus.[10]

Gamma-ray spectra of Re^{187}, obtained with the help of the apparatus described, are shown in Figs. 3 and 4.

MEASUREMENTS AND RESULTS

The single (Fig. 3) and coincidence (Fig. 4) γ-ray spectra of Re^{187} obtained in the present work are completely analogous to the spectra of reference 4. However, the 552-kev γ line, which is not seen in the single spectrum (Fig. 3) and

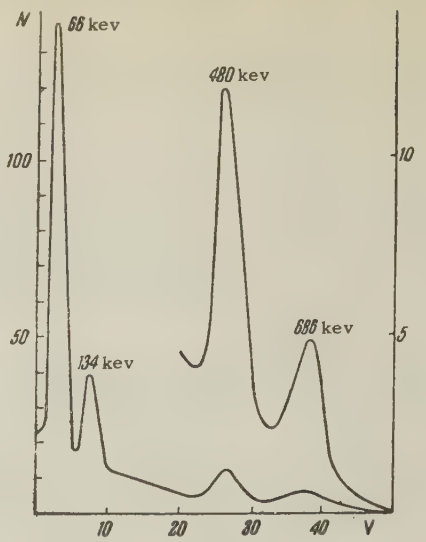

FIG. 3. Single spectra of Re^{187} γ rays (for the upper curve the reading of N is on the right-hand ordinate axis; V is the bias voltage).

did not appear in the coincidence spectra of reference 4, was clearly obtained in the spectrum of instantaneous coincidences with the 134-kev γ line, shown in Fig. 4. From the ratio of the areas of the photopeaks at 480 and 552 kev (Fig. 4), taking into account the experimentally determined[11] efficiency of the NaI crystal, the resolving power of the coincidence circuit of 2τ and the lifetime of the 206 kev energy excited level, a value of 0.10 ± 0.02 was obtained for the ratio of the intensities $I(552)/I(480)$ of the γ-lines, which is four times less than the value obtained for this ratio in reference 6. As is evident from Fig. 4, the slits of the amplitude analyzer could, by suitable adjustment, almost completely separate the

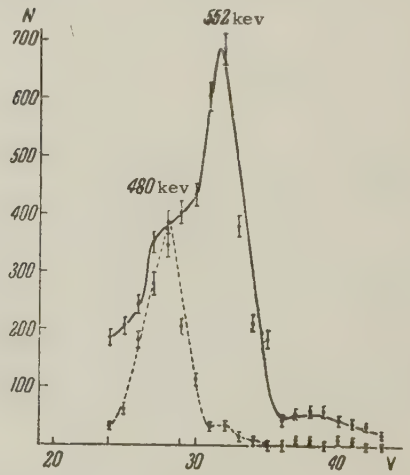

FIG. 4. Section of the γ-coincidence spectrum with 134-kev line in Re^{187}. The solid line is the instantaneous-coincidence spectrum, the dotted, the delayed-coincidence spectrum taken for a delay time $t_d = 0.1$ μsec. The resolving powers of the channels of the coincidence circuit is $\tau = 1.6 \times 10^{-8}$ sec.

552 kev γ line from the 480 kev γ line. This circumstance was used for the determination of the internal-conversion coefficient α_K^{134} in the K shell for the 134-kev γ transition, and also for the measurement of the angular correlation of the 552 — 134 kev γ cascade. The coefficient α_K^{134} was determined from the ratios of the areas of the photopeaks of the x-ray K line of Re and the 134-kev γ line in the coincidence spectrum with the 552-kev γ line. After introducing corrections for the efficiency of the NaI crystal, for the fluorescence yield of Re, and for absorption of the γ and x-radiation in the equipment (the total of all the corrections was 5%), a value $\alpha_K^{134} = 2.0 \pm 0.2$ was obtained.

In the measurements of the angular correlation of the 552 — 134 kev cascade, as in all other measurements, a preparation of W^{187} was used, obtained by the reaction $W^{186}(n, \gamma)W^{187}$. In the majority of cases the target was a tungsten wire of diameter 0.2 mm, from which sources with lengths from one to several millimeters were directly prepared. The distance from the source to the NaI crystal was 70 mm. The 552-kev γ line was registered by the first counter, in which a 20 mm high NaI(Tl) crystal served as a detector, while the slit of the amplitude analyzer was set so that the contribution of the 480-kev γ line did not exceed 5%. The second counter with a 13 mm high NaI crystal registered the 134-kev γ line. As a precaution against Compton scattering from one crystal into the other, which would lead to false coincidences, the NaI crystal of the first counter was covered by a lead filter 3 mm thick in front and 4 mm thick on the side surfaces. Coincidences were registered in a three-channel circuit. Two of these channels served as monitors: in the case of a normally working circuit and accurate set-up of the slit of the amplitude analyzers, these channels could not register any coincidences except random ones. The resolving time of the coincidence channels was determined by random coincidences from independent sources and was equal to 2.5×10^{-8} sec in each channel. The number of coincidences at angles θ of 90, 135, and 180° alternately, with an interval of 10 minutes, was measured with the aim of determining the angular correlation function $W(\theta)$. Simultaneously with the measurement of the coincidences at a given angle, the number of single charges of the counters were determined. About 15,000 true coincidences were registered for each of the foregoing angles, θ, with the ratio of the number of true coincidences to the number of random coincidences equal to $N_t/N_r = 2$. The average true coincidence rate was 9 counts/minute. In calculating $W(\theta)$ corrections were introduced for a variation of the single pulses in the counters, for the finite angular resolution of the detectors, and for the decay of the W^{187} source (half-life $T_{1/2} = 24$ hrs.). The contribution of the high-energy lines was not taken into account. As a result the following was obtained for the angular correlation function of the 552 — 134 kev γ cascade in Re^{187}

$$W(\theta) = 1 + (0.023 \pm 0.014)\cos^2\theta,$$

Which corresponds to the sequence of transitions $\frac{5}{2}(E1)\frac{7}{2}(M1+E2)\frac{5}{2}$, if the ratio of the amplitudes of the mixture $\delta^2 = I(E2)/I(M1)$ of the 134-kev transition is taken equal to $(2.2 \pm 0.5) \times 10^{-2}$, in agreement with the Coulomb excitation data,[5] with $\delta < 0$ (see Fig. 5).

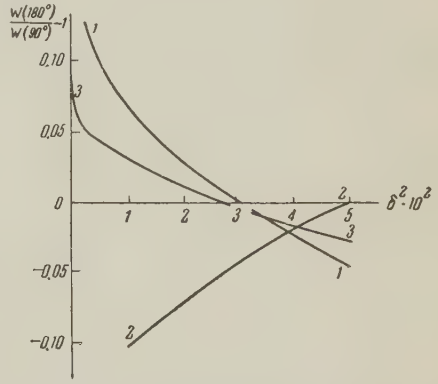

FIG. 5. Curves of W(180°)/W(90°) − 1 (theoretical) for the 552 − 134, 480 − 72 − 134 and 72 − 134 kev γ transitions as functions of ratio of the amplitudes $\delta^2 = I(E2)/I(M1)$ of the E2 and M1 radiation mixture of the 134-kev γ transition for the following sequences of transitions: 1) $\frac{5}{2}$ (E1) $\frac{7}{2}$ (M1 + E2) $\frac{5}{2}$, 2) $\frac{5}{2}$ (E2) $\frac{9}{2}$ (E1) $\frac{7}{2}$ (M1 + E2) $\frac{5}{2}$, 3) $\frac{9}{2}$ (E1) $\frac{7}{2}$ (M1 + E2) $\frac{5}{2}$.

Along with those described above, measurements of the angular correlations of the 480 — 134 and 72 — 134 kev γ cascades in Re^{187} were made. The first measurements of the 480 — 134 kev cascade correlations were made with a five channel delayed-coincidence circuit with the width of each channel equal to $2\tau = 1.7 \times 10^{-7}$ sec. The sixth channel of the circuit registered the zero-delay coincidences. Thus the behavior of the function $W(\theta)$ of the indicated cascade could be studied in its dependence on the delay time over the interval from 2×10^{-7} to 10^{-6} sec. The measurement conditions were similar to those described earlier, the only difference being that no measures were taken to protect against scattering from crystal to crystal, since this effect does not give delay coincidences. Besides, the measurements were made not only with metallic, but with liquid sources, aqueous solutions of ammonium tungstate.

The angular distribution of the 480 — 134 kev γ cascade for all five channels of delayed coincidences turned out to be isotropic within 1 to 2%, both in the case of solid and in the case of liquid sources. Inasmuch as this fact can be explained by disruption of the angular correlation, owing to the interaction between the Re^{187} nucleus in the intermediate state and the fields external to the nucleus,[12] the measurements were repeated using only one channel with instantaneous coincidences. Here the resolving power τ of this channel was decreased to 1.5×10^{-8} sec, and for protection from scattering of radiation from crystal to crystal a lead filter was again used. The true coincidence counting rate in this series of measurements consisted on the average of 4 counts/minute with an N_t/N_r ratio $\approx$ 1.3. For each of the three angles θ (90, 135, and 180°), about 8000 true coincidences were obtained. After correcting for the finite angular resolution of the detectors, that the coefficient a_2 was found to equal -0.04 ± 0.02, if the correlation function has the form $W(\theta) = 1 + a_2 \cos^2\theta$. If we take into account in the measurements the contribution of the 552-kev γ line (this contribution amounts to $(35 \pm 5)\%$ of the contribution from the 480 kev γ-line), then a_2 is found to be -0.06 ± 0.04. This result agrees best of all with the data of the measurement of the 552 — 134 kev cascade in the case of the transition sequence $5/2\,(E2)\,9/2\,(E1)\,7/2\,(M1+E2)\,5/2$. Here the ratio of the amplitudes of the mixture in the last transition (134 kev) $\delta^2 = (2.5 \pm 1.5) \times 10^{-2}$; $\delta < 0$ (Fig. 5).

If the foregoing arrangement of moments and multipolarities are correct, the angular correlations of the 72 — 134 kev γ cascade must correspond to the transition sequence $9/2\,(E1)\,7/2\,(M1+E2)\,5/2$. The measurement results turned out to be in agreement with this conclusion. It is impossible to use a frontal lead filter for protection against scattering from crystal to crystal in the measurement of the angular correlation of the 72 — 134 kev cascade. Therefore only one lead side screen was used, but the measurements were made for θ equal to 30, 45, 60 and 90° in such a geometry that the side screen completely excluded the possibility of registering quanta scattered from crystal to crystal. The resolving time of the coincidence circuit was 1.5×10^{-7} sec, and the distance from the source to the NaI crystal was 100 mm. Under these conditions a value 0.009 ± 0.010 was obtained for the coefficient a_2. In the separation of the 72-kev photopeak, we registered the K-shell x-rays from Re, formed as a result of internal conversion. The radiation due to these coincidences possesses an isotropic angular distribution. Knowing the conversion coefficients for the corresponding transitions, we can calculate the contribution of the x-rays to the measured correlation. In the case of the sequence of transitions $9/2\,(E1)\,7/2\,(M1+E2)\,5/2$ the 72-kev γ transition is E1. Utilizing the theoretical values for the K- and L-shell internal conversion coefficients, the $\alpha_K = 0.5$ (reference 13) and $\alpha_L = 0.14$,[14] and also taking into account the contributions of the 480 and 552 kev γ lines in a manner similar to that used in reference 7, we find $a_2 = 0.015 \pm 0.017$.

We obtain therefore for the ratio of the 134-kev transition amplitudes $\delta^2 = (1.7 \pm 0.7) \times 10^{-2}$; $\delta < 0$ (see Fig. 5).

CONCLUSION

The results of the present work confirm the assignment of the angular momenta of the excited states and the multipole types of the Re^{187} γ transitions proposed in the decay scheme of reference 15. According to this scheme, the 206-kev metastable state of Re^{187} has a spin $3/2$ and negative parity. In reference 7 a spin of $9/2$ and positive parity were ascribed to the indicated state on the basis of measurements of the angular correlation of the 72 — 134 kev γ cascade. However, it seems unlikely that a 206-kev state possessing the same parity as the ground state and differing from the latter in momentum only by unity would have so long a lifetime.

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POLARIZATION OF COBALT AND IRON NUCLEI IN FERROMAGNETS

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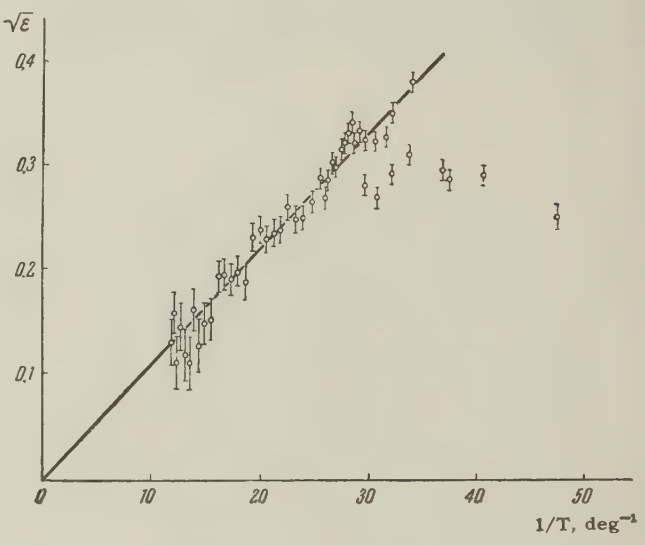
The anisotropy of gamma rays from Co^{60} in a magnetized cobalt-iron alloy (Permendur) was measured at temperatures from 0.03 to 0.1° K. The effective magnetic field strength $H = 2.5 \times 10^5$ gauss was obtained at the cobalt nucleus. No gamma-ray anisotropy was detected in similar experiments on Fe^{59} nuclei in Armco iron cooled down to 0.02° K.

A few years ago Khutsishvili suggested a procedure for polarizing nuclei in ferromagnets. In 1955 and 1956 he reported[1,2] the results of an experimental test of this procedure by Alekseevskiĭ, Shchegolev and Zavaritskiĭ,* who observed about 10 — 15% anisotropy of γ rays from Co^{60} at 0.05 — 0.08° K. Similar results were obtained at about the same time by the Kurti group at Oxford.[3] The present note concerns experiments on the polarization of Co^{60} and Fe^{59} nuclei in ferromagnetics.

A specimen of Permendur, a ferromagnetic polycrystalline alloy (50% Co and 50% Fe), of 3 mm diameter and 0.2 mm thickness, was irradiated by thermal neutrons in a reactor. After irradiation the specimen was vacuum annealed and soldered to the end of a copper "cold conductor;" it was then placed in a field of 1000 oersteds between the poles of a small permanent magnet in a cryostat. The specimen was then cooled to a temperature of a few hundredths of a degree by the adiabatic demagnetization of potassium chrome alum. The magnetic temperature was measured with an ac bridge at 60 cps.

γ rays from Co^{60} were registered by two scintillation counters with CsI crystals of 40 mm diameter and 40 mm thickness. The radioactivity of the specimens was 3 or 4 μcuries, which yielded more than 10,000 scintillation counts in 100 sec (at a single temperature). Several series of measurements were obtained with two specimens.

The most reliable experimental results are shown in the figure, where $\epsilon = 1 - N(0)/N(\pi/2)$ is the anisotropy, $N(0)$ is the count from the counter placed parallel to the magnetic field, $N(\pi/2)$ is the count perpendicular to the magnetic field and T is the absolute temperature of the salt. An insignificant correction for heating due to the absorption of beta particles was neglected. Values obtained for the anisotropy down to 0.04° K are well fitted by the straight line



$$\varepsilon = 1.2 \cdot 10^{-4} T^{-2}.$$

The deviation from this line at lower temperatures is associated, in our opinion, with the maximum of the nuclear specific heat of cobalt, which prevents cooling of the specimen to the temperature of the salt.

A comparison of our results with those in reference 4 shows that the field at a cobalt nucleus in the alloy lattice is close in value to the field at an ion in a pure cobalt crystal. In our case $H = 2.5 \times 10^5$ gauss, which corresponds to $A = 2.4 \times 10^{-2}$ ° K for the hyperfine splitting factor. In reference 4 $H = 2.3 \times 10^5$ gauss and $A = 2.2 \times 10^{-2}$ ° K.

The experiment on the polarization of Fe^{59} nuclei was similar to that performed with cobalt. The specimen of Armco iron was a disk 3 mm in diameter and 0.1 mm thick, which was annealed

*The data obtained by N. V. Zavaritskiĭ are given in greater detail in reference 2.

after being irradiated in a reactor. The initial activity of the disk did not exceed 2 or 3 μcuries. The counters registered only gamma rays with 1290 kev (from the transition $^3/_2^- \xrightarrow{\gamma} {}^7/_2^-$) emitted after $^3/_2^- \xrightarrow{\beta} {}^3/_2^-$ beta decay.

In the temperature range from 0.02 or 0.03 to 1° K these experiments detected no difference between the gamma-ray counts parallel and perpendicular to the field, with better than 0.5% accuracy. This result disagrees with that given in reference 5. Assuming that the field of the electron shells at the Fe^{59} nucleus does not differ greatly from that at the cobalt nucleus, we can obtain an upper limit for the magnetic moment of the Fe^{59} nucleus. With $H = 2.5 \times 10^5$ gauss and $\epsilon \lesssim 0.5\%$ we obtain $\mu(Fe^{59}) \leq 1.5$ nuclear magnetons.

The authors wish to thank E. K. Zavoĭskiĭ for his continued interest and valuable suggestions, and L. V. Groshev for a discussion of the results.

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THE INFLUENCE OF TEMPERATURE AND PRESSURE ON THE RAMAN SPECTRUM OF NITROGEN

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The rotational structure of the Q branch of the vibrational band of the Raman spectrum of nitrogen was studied. The dissimilar nature of the impact broadening of the vibrational and rotational lines of the spectra is established. The following effective collision radii determining the Raman spectrum line broadening were found: for the Q branch of the vibrational band $\rho_\nu = 0.43$ A, for the purely rotational band lines $\rho_\omega = 3.9$ A. It is shown that the experimental results do not contradict the impact theory of spectral line broadening.

THE phenomenon of Raman scattering is fundamentally related to the intramolecular vibrations and rotations of molecules. However, the intermolecular interaction also leaves its imprint on these processes and is reflected in the spectrum. Therefore an investigation of the spectra makes possible the study of the influence of the intermolecular interaction on the processes related to Raman scattering and, besides, allows a more complete disclosure of the dynamics of molecules in the presence of an external perturbation.

The study of the manifestation of the intermolecular interaction in the spectrum begins logically with a gas at moderate pressures, whose kinetics of motion are well known, and therefore the treatment of results is facilitated. Besides, such a study may turn out to be useful in the investigation of phase transitions and the structure of liquids.

The effect of the intermolecular forces is small, and affects mostly the line contours. An investigation of the line contour, as far as the theory of collision broadening is concerned, makes it possible to find the effective optical collision radii for the processes that determine the Raman-scattering line broadening.

Measurement of the line widths involves considerable experimental difficulties, and therefore comparatively little work has been done on intermolecular interactions in Raman scattering. The most interesting in this direction is the work of Sterin.[1] He measured the line widths of the completely symmetrical vibrations of benzene ($\Delta\nu$ = 992 cm^{-1}) in the liquid and in the vapor. It turned out that the line width was not changed in the phase transition, a fact explained by the rotational splitting of the Q branch of the vibrational band.[2]

The aim of the present work was to investigate the line broadening of Raman scattering in a gas, to study in more detail the rotational splitting of the vibrational-transition Q branch, and to study the spectrum of a liquid. Nitrogen was chosen as the object of the investigation.

EXPERIMENT

A high-pressure illuminator for the investigation of Raman scattering in gases at various pressures and constant temperature was developed and manufactured jointly with the High-Pressure Physics Institute of the U.S.S.R. Academy of Science; its construction is shown in Fig. 1. Three PRK-2 lamps, surrounded by a running-water cooling filter, were placed parallel to the scattering volume and illuminated it through cylindrical lenses of organic glass. The lenses, placed in a "hot" setting, served simultaneously as seals for the illumination windows. The illuminator withstood a test pressure of 450 atmos. Within the scattering volume was placed a multiply reflecting system with aluminized mirrors, which permitted a shortening of the exposure time by a factor of six

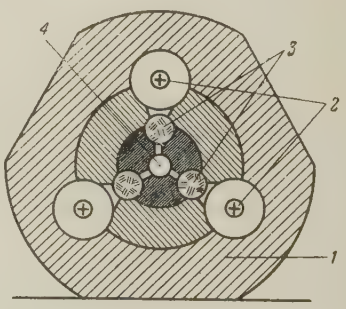

FIG. 1. Diagram of the high pressure illuminator: 1 – housing; 2 – illuminating lamps with water filters; 3 – cylindrical lenses 4 – scattering volume.

or seven.[3] With the help of a specially-designed double-lens condenser, the forward aperture of the mirror system was reflected on the spectrograph aperture, and the rear of the mirror was reflected on the collimator. The gas in the illuminator was supplied directly from cylinders.

A double-lamp illuminator with cylindrical lenses and water-cooled mercury lamps was used to photograph the liquid nitrogen.[4]

The Raman scattering spectrum was excited by the 4047 and 4358 A mercury lines and recorded on a HUET V-III spectrograph with a linear dispersion equal to 22.7 cm^{-1}/mm in the 4047-A region and to 36.2 cm^{-1}/mm in the 4358-A region. The exposure time of the Raman-Ortho Agfa plates ranged from 3 to 40 hours. The spectrograph was thermostatically controlled during the exposure time.

Each experimental point for a given pressure was obtained as an average from three to four photographs. The photographs were processed on an MF-4 microphotometer; the total width of the excitation line, of the slit, and of the apparatus function of the spectrograph and the emulsion (~ 1.5 cm^{-1} for the gas and ~ 1 cm^{-1} for the liquid) was excluded from the results of the measurements[8] in the subsequent processing. The average error in the measurements of the line widths comprised about 10%.

The Raman scattering spectrum of nitrogen (gas) consists of O and S branches of rotational bands situated on both sides of the excitation line, and a Q branch of the vibrational band. The scattering spectrum in nitrogen was studied at 27° C in the pressure interval from 7 to 114 atmos.

The rotational spectrum for low pressures consists of sharp equidistant lines of alternating intensities with component distances of 8.0 cm^{-1}; up to 15 lines of the O branch and 25 lines of the S branch were observed. On increasing the pressure, the lines of the rotational spectrum broadened; at a pressure of 20 atmos they began to overlap and for $p > 80$ atmos they blended into a continuous band with weakly defined maxima at its width in the region of pressure investigated the positions of the intense rotational lines (Fig. 2).

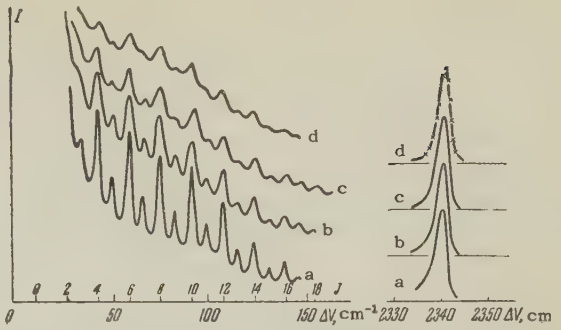

FIG. 2. Raman scattering spectra of nitrogen at pressures a) 15, b) 25, c) 40 and d) 60 atmos; t = 27° C.

Only the Q branch of the vibrational-rotational transition is observed in the vibrational band of nitrogen ($\Delta\nu = 2340$ cm^{-1}); the O and S branches, being weaker in intensity, were not observed on the photographs.

The Q branch of the vibrational band of nitrogen is narrow (~ 2 cm^{-1}) and has an asymmetric form with an arm on the violet side. On increasing the pressure the Q branch broadens slightly, being less than the width of the pure rotational band component.

The variation with pressure of the rotational ($\delta\omega$) and vibrational ($\delta\nu$) Raman scattering line widths of nitrogen is shown in Table I and in Fig. 3; the average width for the transitions with $J = 0 - 15$ is given for the rotational lines.

To clarify the role of the temperature, the spectrum of nitrogen was investigated by us at t = 200° C (pressure p = 90 atmos, reduced to 27° C) and at the temperature of liquid nitrogen, at t = −190° C. The investigation showed that the nitro-

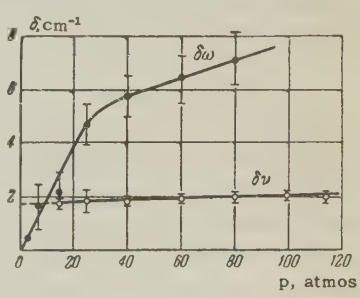

FIG. 3. Pressure dependence of the widths of the rotational, $\delta\omega$, and the vibrational, $\delta\nu$, lines of the nitrogen Raman scattering.

TABLE I. Dependence of the nitrogen Raman scattering line width on pressure (t = 27° C)

p, atmos	δω, cm^{-1}	δν, cm^{-1}	p, atmos	δω, cm^{-1}	δν, cm^{-1}
3	∼0,5*	—	60	6,4±0,9	1,9±0,2
7	1,75±0,9	—	80	7,1±1,0	1,95±0,2
15	2,2±0,7	1,7±0,4	100	—	2,1±0,2
25	4,7±0,7	1,85±0,5	114	—	2,4±0,3
40	5,8±0,8	1,8±0,2	—	—	—

*Measured by V. A. Iozenas, Moscow State University.

gen lines are wider at 200° C than at room temperature: $\delta\omega = 8.5 \pm 2$ cm^{-1}, $\delta\nu = 5.4 \pm 1.5$ cm^{-1}. Only the Q branch of the vibrational band is observed in the liquid nitrogen spectrum with a width $\delta\nu = 0.4 \pm 0.2$ cm^{-1}, the rotational bands being absent.[6]

The frequency shift of the Raman scattering lines lies within the limits of the errors of measurement (~ 1 cm^{-1}).

DISCUSSION OF RESULTS

1. The broadening due to molecular collisions,[1,2] the analysis of which is best begun with an investigation of the pressure dependence at constant temperature, is the basic cause of the observed line broadening.

For the rotational lines in the region investigated the pressure dependence $\delta\omega(p)$ has the following form: from 0 to approximately 25 atmos it is a straight line, at 25 atmos there is a bend, then the straight line begins anew, but with smaller slope (see Fig. 3). According to the collision-broadening mechanism, this motion of the curve may be explained in the following manner. In the 0 — 25 atmos region the density of molecules is comparatively small, the collision duration is much less than the time between collisions, and the levels do not yet overlap one another, therefore the theory of collision broadening can be applied here. This allows us to estimate the effective optical-collision radius ρ_ω for the rotational levels. From collision theory, the line width is $\delta = 2\pi N\bar{v}\rho^2$ (N is the number of molecules per cm^3, $\bar{v}$ is the average molecular velocity[7]). For nitrogen at 27° C this relation leads to the following dependence of the line width on the pressure:

$$\delta = 0{,}027\ \rho^2 p$$

(ρ is in A, p is in atmos, and δ is in cm^{-1}). Utilizing the experimental dependence for $\delta\omega(p)$, we find the effective collision radius for the rotational lines of nitrogen $\rho_\omega = 3.3$ A. The value of ρ_ω allows us to estimate the upper bound of the region of the collision broadening influence and, correspondingly, the beginning of the statistical broadening region. The collision theory is accurate[7] for $2\pi\rho^3 N \ll 1$, whence the upper bound of the pressure for collision broadening of the rotational levels is $p \ll 100$ atmos.

In the region above 25 atmos the rotational levels overlap, and the slope of the curve becomes smaller. A systematic error in the calculation of the width arises for strongly overlapping lines, i.e., the calculation does not take into account the drop in line intensity at the edge of the rotational spectrum; in addition, the broadening mechanism for the overlapping levels is not completely clear; this fact may be the cause of the change in slope of the curve $\delta\omega(p)$.

Besides the collision broadening, the unresolved rotational structure enters into the observed width for the Q branch of the vibrational band of nitrogen. From the $\delta\nu(p)$ curve extrapolated to zero pressure we can separate out the constant part of the width due to the rotational splitting; the remaining part of the width, which is pressure dependent, naturally belongs to the collision broadening. The magnitude of the rotational splitting coincides with the calculations carried out below, the linear variation of the collision broadening agrees with the theory, and therefore the separation operation is completely admissible.

The width of the Q branch of nitrogen depends weakly on the pressure. Calculations based on collision theory and utilizing the experimental data yield $\rho_\nu = 0.43$ A for the effective optical collision radius for the vibrational line, i.e., significantly less than the kinetic-theory radius, $\rho_{gas} = 1.84$ A. Such a small value of ρ_ν signifies only a small role of collisions in the broadening of the Q branch. The different values of ρ_ν and ρ_ω observed in practice testify to the different mechanisms of collision broadening of the rotational and vibrational lines. Here, evidently, the following fact plays a role: unlike the O and S branches of the rotational spectra (transitions $\Delta\nu = 0$, $\Delta J = \pm 2$), the Q branch is composed of transitions in which the rotational state does not change ($\Delta J = 0$). Therefore only the perturbation of the vibrational component of the Q-branch structure during a collision influences the width, and perturbation of the rotational components does not manifest itself.

The effective radii found for the nitrogen Raman scattering, $\rho_\omega = 3.3$ A and $\rho_\nu = 0.43$ A, differ from the radii for the nitrogen microwave band $\rho_{mw} = 1.4 \sim 2.25$ A, quoted in reference 8; the observed difference points out the different effectiveness of the impact for these processes.

2. It was indicated above that the observed width of the Q branch was not due solely to collision broadening. The change of the moment of inertia of the molecules in vibration changes the separation between the rotational levels, and the rotational components of the Q branch are not exactly superimposed one on another, but are somewhat split. This structure, not resolved by the apparatus, introduces an additional contribution into the observed width.

The frequencies of the rotational component of the Q branch are described by the formula

$$\nu = \nu_0 - 2x_0\nu_0 - \alpha J(J+1),$$

where $\nu_0 - 2x_0\nu_0$ is the frequency of the undisplaced component, and $\alpha = B_0 - B_1$ is the vibrational-rotational interaction constant. The distribution of intensities among the different components is given by the expression (see reference 2)

$$I_\nu = I_0 \sqrt{1 + 4\nu/\alpha}\; e^{-B\nu/kT\alpha}.$$

The rotational structure of such a type is shown dotted in Fig. 4. Taking into account the dependence on J of the distance between components (which was neglected in reference 2), it is easy to find the averaged intensity distribution in the Q branch:

$$I_{av} = I_0 \exp\left\{-\frac{B}{kT}\frac{\nu}{\alpha}\right\} \Big/ [1 + (1 + 4\nu/\alpha)^{-1/2}],$$

(the cross-hatched contour in Fig. 4). Graphical calculation for N_2 ($B = 1.998\ cm^{-1}$, $\alpha = 0.171\ cm^{-1}$ and $T = 300°K$) leads to a width due to the unresolved rotational structure $\delta\nu_{rot} = 1.54\ cm^{-1}$.

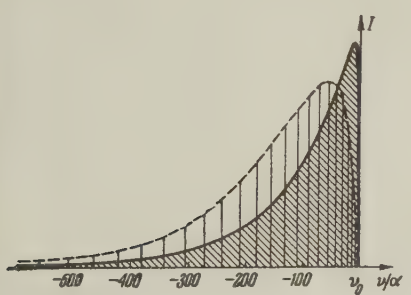

FIG. 4. The rotational structure of the nitrogen Q band at 300°K.

The pressure-independent part of the width, which is separated from the observed dependence of $\delta\nu(p)$ by extrapolating the curve to zero pressure, equals $1.6\ cm^{-1}$, i.e., it coincides with the calculations, and hence is explained by the rotational structure of the Q branch unresolved by the apparatus.

The asymmetry of the observed contour serves as additional proof of the presence of rotational splitting. For p = 60 atmos, each rotational component of the Q branch was broadened to the magnitude of the collision and apparatus broadening ($\sim 1.5\ cm^{-1}$) and a rounded contour was constructed which coincided well with the one observed (in Fig. 2 calculated points are denoted by crosses).

As is evident from Table II, the rotational splitting introduces a basic length into the observed width of the Q branch and a significantly larger collision broadening over a comparatively wide interval of temperature.

3. In conclusion we shall dwell momentarily on the role of the temperature. The frequency and energy of collisions and the distribution of the molecules over the rotational levels changes with the temperature; with an increase in temperature both these factors operate in identical directions: in view of the change of the populations of the rotational levels of the Q branch, more and more components appear and the width of their structure grows, the collision broadening of the rotational and vibrational lines also increasing.

A detailed examination of the rotational structure of the Q branch and of the value obtained for ρ_ν found allows us to calculate the width of the vibrational lines at various temperatures; in Table II the present calculations are compared with the results of measurements for nitrogen at t = 200, 27 and −190°C. The measured values of $\delta\nu$ coincide with the calculations within the limits of error. For −190°C the experimental data belong to the liquid, whereas the calculations were made for the gas. The agreement between the present calculations and the observed width in the liquid may be due to the width being fundamentally caused by rotational splitting, which is also present in the liquid; only a small part is due to the range of the intermolecular interaction.

The present work was conducted at the suggestion of the prematurely deceased Academician G. S. Landsberg, and was conducted under his direction in the first stages. I consider it my duty to thank Profs. P. A. Bazhulin and I. I. Sobel'man for advice in the performance of the work and the discussion of results, and also my associates at the High-Pressure Physics Institute

TABLE II. Comparative contributions of the collision broadening and rotational splitting to the width of the Q branch of nitrogen at various temperatures

t, °C	p at 27°C	Calculated for the gas			Experimental
		$\delta\nu_{rot}$	$\delta\nu_{col}$	$\delta\nu=\delta\nu_{rot}+\delta\nu_{col}$	$\delta\nu$, cm^{-1}
200	50 atmos	2,34	1,4	3.8	5.4±1.5
27	50 atmos	1,54	0,2	1,75	1.8±0.2
—190	liquid, $\rho = 0.5\ g/cm^3$	0,45	0,05	0.5	0.4±0.2

of the U.S.S.R. Academy of Sciences, Profs. L. F. Vereshchagin, E. F. Shcherbakov, and I. E. Surkov, for help in the construction and preparation of the high-pressure illuminator.

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PRODUCTION OF Mo^{93m} *IN THE REACTION* $Se^{80}(O^{16}, 3n)$

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The isomer Mo^{93m} has been obtained by irradiating the separated isotope Se^{80} with O^{16} ions accelerated in a 150-cm cyclotron. The isomer results when three neutrons evaporate from the compound nucleus produced by the complete coalescence of Se^{80} and O^{16} nuclei. The excitation function of this reaction and the absolute cross section for Mo^{93m} production have been measured.

IT has been established in a number of investigations[1-5] that the excitation energy of the isomer Mo^{93m} is 2460 kev and that its half-life is 6.95 hrs. The isomeric nucleus drops to its ground level by the successive emission of 270-, 690-, and 1490-kev gamma rays. Mo^{93m} cannot be produced very easily; it was not detected after the bombardment of Mo^{92} with 10-Mev deuterons[1] or thermal neutrons,[3,4] or after the irradiation of Mo^{94} by 23- and 32-Mev γ rays.[2,4] The reaction $Mo^{94}(n, 2n)Mo^{93m}$ has a small cross section. The data in references 1 — 3 show that the cross section for $Mo^{92}(d, p)Mo^{93m}$ with $E_\alpha = 10$ Mev and that for $Mo^{92}(n, \gamma)Mo^{93m}$ are smaller than 10^{-30} cm^2, and that the cross section for $Mo^{94}(n, 2n)Mo^{93m}$ is of the order of 10^{-29} cm^2.

The cross section for the reaction $Nb^{93}(p, n)Mo^{93m}$ with 6.7-Mev protons is 2×10^{-27} cm^2, which is considerably smaller than the cross sections for the production of the isomeric and ground states by means of the indicated (p, n) reaction in neighboring nuclei.[2]

According to Goldhaber[6] and to Kraushaar[7] the spin of Mo^{93m} is $^{23}/_2$, although the value $^{21}/_2$ is suggested in reference 8. This large spin evidently accounts for the small cross sections for the production of Mo^{93m} in the reactions that have been mentioned.

Reactions with heavy ions at ~ 100 Mev result in the formation of compound nuclei whose angular momentum may be as high as ~ 50. The evaporation of neutrons cannot essentially reduce the angular momentum of the compound nucleus; we can therefore expect that a relatively large yield of Mo^{93m} will result from reactions with heavy ions.

In the present work Mo^{93m} was produced by the reaction $Se^{80}(O^{16}, 3n)$. For this purpose we used a procedure developed at the Atomic Energy Institute of the U.S.S.R. Academy of Sciences for studying nuclear reactions induced by multiply-charged ions accelerated in a 150-cm cyclotron. Stacks of aluminum foils bearing hot pressed layers of 92% enriched Se^{80} were bombarded in the internal cyclotron beam by monoenergetic quintuply-charged oxygen ions at 102 ± 2 Mev. The oxygen ion current was ~ 0.1 μa and the irradiation time was 1 to 1.5 hrs. γ rays from the isomer Mo^{93m} were analyzed by means of a scintillation spectrometer with a NaI(Tl) crystal, which had been calibrated by means of well-known lines of Hf^{181} (56, 135, 340, and 480 kev), Cs^{137} (662 kev) and Co^{60} (1170 and 1320 kev). The spectral sensitivity was determined by two methods, the first of which employed Cu^{61} γ rays from 70 to 1200 kev, whose relative intensities are given in reference 9. The γ-ray spectrum was measured under exactly the same conditions as the main experiment. The β activity of the specimens was measured at the same time by a calibrated β-ray counter. Through a comparison of the γ-ray and β-particle intensities from Cu^{61} we were able to determine the relative spectral sensitivity of the spectrometer and its absolute efficiency for the registration of γ rays. Cu^{61} was obtained through the reaction $V^{51}(C^{13}, 3n)Cu^{61}$, which we have investigated previously.[10]

In the second method the number of β particles from Cs^{137}, measured by a β-ray counter with 4π solid angle, was compared with the number of 662-kev γ rays registered by the spectrometer. The results obtained by the two methods were in agreement.

One of the γ-ray spectra is shown in Fig. 1; 270-, 690-, and 1490-kev γ rays are seen to result from the bombardment of Se^{80} by O^{16} ions. These lines have a half-life of 7 hrs and the fact that they belong to a molybdenum isomer was verified through the chemical separation of molybdenum from the selenium target. β activity with the same half-life at about 300 kev was also detected;

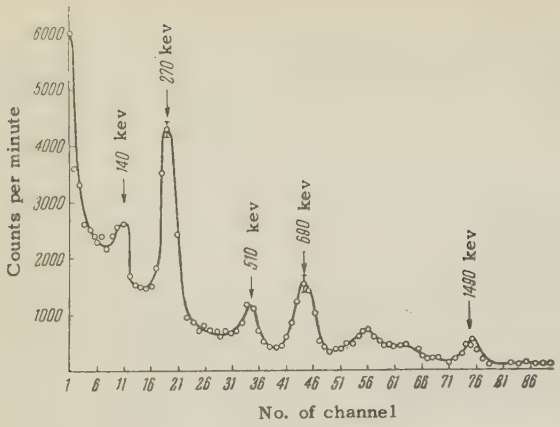

FIG. 1

this is evidently due to internal conversion electrons from the decay of Mo^{93m}. The chemical analysis, investigation of the γ-ray spectrum and measurement of the half-life provide the basis for assigning the observed activity to Mo^{93m}.

The dependence of the cross section for $Se^{80}(O^{16}, 3n)Mo^{93m}$ on the energy of the oxygen ions was studied in two experiments. In the first of these experiments, represented by the solid lines in Fig. 2, a stack of five foils was irradiated. A 3 mg/cm^2 layer of Se^{80} was deposited on each 5-micron aluminum foil. In the second experiment, represented by the dashed lines in Fig. 2, the stack consisted of four 2-micron foils with a 7 mg/cm^2 layer of Se^{80}.

Figure 2 shows that the energy dependence of the cross section for Mo^{93m} production has the form of the typical excitation function of reactions that proceed via a compound nucleus accompanied by competing processes. The maximum cross section for 70-Mev oxygen ions is 240 mbn with a possible error of 50%. When the oxygen ion energy is ~100 Mev the cross section does not exceed 5 mbn.

These experiments show that nuclear isomers can be produced efficiently by means of heavy ions, especially those isomers which possess large spin.

In performing this work we were constantly assisted by the valuable suggestions of G. N. Flerov and by the experimental procedures for work with multiply-charged ions that have been developed by the group which he directs.

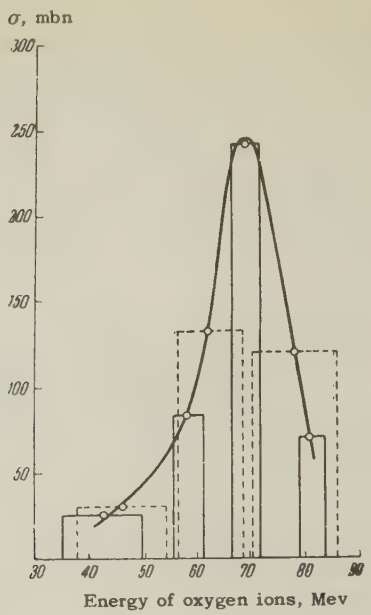

FIG. 2

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ANGULAR DISTRIBUTIONS IN THE REACTIONS Ne^{22} *(d, p)* Ne^{23} *AND* A^{36} *(d, p)* A^{37}

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Thick nuclear emulsions were used to study the angular distributions of protons from (d, p) stripping reactions initiated by 4-Mev deuterons in gaseous targets enriched with Ne^{22} and A^{36}. It was established that the orbital angular momenta of the neutron captured into the ground and first (0.98 Mev) excited states of Ne^{23} are 2 and 0 (shell model configurations $(1d_{5/2})^{-1}$ and $(2s_{1/2})^{-1}$), while in A^{37} the orbital angular momentum of the neutron in the ground state is 2 [corresponding to a configuration $(1d_{3/2})^{-1}$].

INTRODUCTION

PARITIES and probable values for spins of the final nuclei in stripping reactions can be determined from studies of the angular distributions.[1,2] As is well known, the Coulomb and nuclear interactions are not taken into account in the Butler theory. These factors have little effect on the positions of the peak in the angular distribution, although the Coulomb interaction does move the peak a little toward bigger angles while the nuclear interaction has the opposite effect. Up to $Z \approx 20$ and for deuteron energies the same order of magnitude as the Coulomb barrier in the target nucleus, the orbital angular momentum l_n of the captured particle can usually be inferred from the position of the peak in the angular distribution. However, the Coulomb, and to a lesser extent the nuclear, interactions have a big influence on the cross section.

The Coulomb interaction is hard to take into account exactly[3] because the calculations are complicated. Taking the Coulomb and nuclear interactions into account, the reduced widths Θ_l^2 of single particle states in the final nucleus should be little less than one, if measured in units of Wigner's sum rule limit, while $\Theta_l^2 \ll 1$ for states with a complicated structure, formed by many-particle excitation. Hence the value of Θ_l^2 obtained from such experiments without taking into account corrections has, essentially, only qualitative significance. However, if two-lying states of the final nucleus have approximately equal values of Θ_l^2, then, considering the corrections due to Coulomb and nuclear forces to be of the same order of magnitude, that is evidence that the two states have similar types of excitation.

In the work being reported upon here, the angular distributions in (d, p) stripping reactions occurring in gaseous targets enriched with the isotopes Ne^{22} (enriched by 90%) and A^{36} (enriched by 11%). The angular distribution associated with the reaction Ne^{22} (d, p) Ne^{23} has been studied by several authors (for example, reference 4); in the case of the reaction A^{36} (d,p) A^{37}, the angular distribution is reported for the first time.

DESCRIPTION OF THE EXPERIMENT

Experimentally, it is harder to study the angular distributions from gaseous targets than it is from solid ones, especially when nuclear emulsions are used as detectors, since in this case it is necessary to define the directions of the outgoing particles simultaneously for many angles. The best way of doing this is to use the target construction described by Chadwick, Burrows, and others,[5,6] where the ring-shaped space between two concentric cylinders forms a common window for all angles. A drawback of this construction is that a large volume of gas is needed, filling the whole chamber with target and emulsions. We had only a small amount of enriched gas available, so the construction was modified somewhat.

A diagram of the experiment is shown in Fig. 1. Accelerated to approximately 4 Mev by the cyclotron of the Institute of Nuclear Physics, Moscow

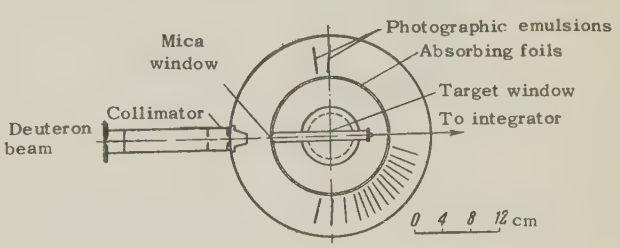

FIG. 1. Schematic diagram of the experiment.

State University, the deuterons passed through a set of collimating slits and were focused on a thin mica window (1.4 mg/cm^2) which limited the volume of gas serving as the target. The particles produced in the reaction passed through two windows (4mm × 4mm) in the sides of the target volume, the windows being covered with thin organic films of thickness 50 μg/cm^2. The total volume of gas required was then the sum of the target volume, the volume of the filling system, and of the pressure gauge. This sum was about 20 cm^3. Under the gas pressure, the film bulged out into the vacuum in the shape of an irregular hemisphere. The products of the reaction passed through the film at approximately right angles for all angles which were counted, so the effect of multiple scattering in the film was held to a minimum. The gas pressures used were about 6.0 cm Hg. Aluminum foils were placed in front of the nuclear emulsions to absorb particles of energies less than those of interest.

The variations in deuteron energy due to different path length in the gas target for various emission angles θ were small and could be neglected (for neon and θ in the range 17 to 60°, $\Delta E_d \approx 0.15$ Mev). The deuteron energy was determined from the range of the most penetrating group of protons and from the known Q of the reaction. In the target gas, multiple scattering did not spread the deuteron beam by more than 30′.

The tracks were scanned with a MBI-2 microscope having a magnification 1.5 × 90 × 5. The errors shown on the angular distributions below (see, for example, Figs. 3, 5) are statistical; the dotted horizontal line shows the isotropic part of the angular distribution. This is presumably connected with formation of the compound nucleus.

EXPERIMENTAL RESULTS AND DISCUSSION

1. Ne^{22} (d, p) Ne^{23}. Figure 2 shows the energy spectra of protons emitted at an angle $\theta_{lab} = 58° 30'$ in (d, p) reactions on Ne^{22} and Ne^{20}. The resolution is not good enough to distinguish between the groups $Ne^{23}(0)$, $Ne^{21}(2)$ and $Ne^{23}(1)$, $Ne^{21}(3)$. However, the groups from reactions on Ne^{20} must be small in view of how rich the sample was in Ne^{22}. Hence the intensities of the proton groups $Ne^{23}(0)$ and $Ne^{23}(1)$, corresponding to transitions to the ground and first excited states of Ne^{23}, were obtained from the normal distribution of track lengths in the histograms. Figure 3 shows the experimental angular distributions, together with theoretical curves calculated using Butler's formulae. The calculations were made with values $R = 6.1 \times 10^{-13}$ cm and $R = 6.5 \times 10^{-13}$ cm for the radii of the ground and first excited states respectively. The angular distribution of the proton group Ne^{23} (0) corresponds to capture of a neutron with orbital angular momentum $l_n = 2$, while that of Ne^{23} (1) corresponds to $l_n = 0$. These results agree with those given in reference 4, the latter having been obtained by magnetic analysis at deuteron energies $E_d \sim 8$ Mev; our values of R were taken from this paper.

The possible values for the spin and parity of the ground state of Ne^{23} are therefore 5/2+ or 3/2+, while the corresponding quantities for the first excited state are 1/2+. The value 5/2+ for the ground state is predicted by the shell model and was used in calculating the reduced width. At the maxima of the angular distribution (and neglecting the isotropic part), the differential cross sections for transitions to the ground and first

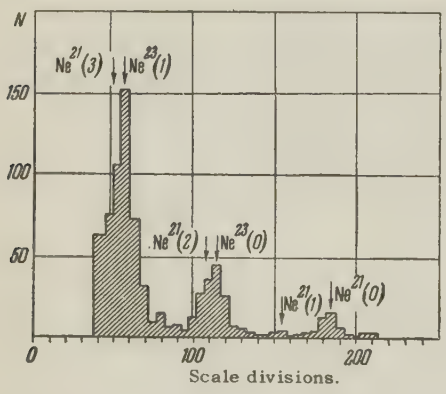

FIG. 2. Proton spectrum from the reactions $Ne^{20,22}$ (d, p) $Ne^{21,23}$ at an angle $\theta_{lab} = 58°30'$. One scale division = 1.25 μ; thickness of the aluminum absorber: 39 mg/cm^2.

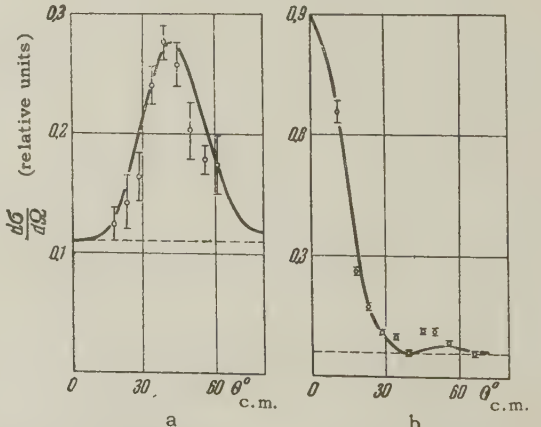

FIG. 3. Angular distributions of protons from the reaction Ne^{22}(d, p)Ne^{23}; a – ground state, $l_n = 2$, b – first excited state $l_n = 0$. $d\sigma/d\Omega$ in units of 12 mbn/sterad.

excited states are 1.87 (38°) and 11.1 (0°) mbn/steradian respectively, while the corresponding values of Θ_l^2 are 0.22×10^{-2} and 0.16×10^{-2}. According to the shell model, the ground state of Ne^{23} with T = 3/2 has the configuration $(1d_{5/2})^{-1}$, corresponding to a "hole" in the filled neutron sub-shell $1d_{5/2}$, while the first excited state (0.98 Mev) is $(2s_{1/2})^{-1}$.

2. A^{36} (d, p) A^{37}. The proton spectra from reactions on the isotopes A^{36}, C^{13} and N^{14} at and angle $\theta_{lab} = 32° 30'$ are shown in Fig. 4. The last two isotopes were present in the target gas as impurities. The statistics are unsatisfactory because there was relatively little A^{36} present, and in addition air as a contaminant; the emulsions were overloaded with grains from the γ-background and tracks from (d, n) reactions on light nuclei (C^{12}, O^{16}). The angular distribution corresponding to transitions to the ground state of A^{37} is shown in Fig. 5. The theoretical curve is calculated from Butler's formulae with $R = 6.7 \times 10^{-13}$ cm. The absolute value of the cross section could not be calculated because the amount of air in the gas was unknown. The results show that a neutron is captured into the ground state of A^{37} with orbital angular momentum $l_n = 2$. Hence the possible values of spin and parity for this state are 3/2+ and 5/2+. The nucleus A^{37} has two proton and one neutron "holes" in the filled $1d_{3/2}$ shell, which according to the shell model implies a configuration $(1d_{3/2})^{-1}$ for the ground state. This conclusion is supported by analysis of the allowed β decay $A^{37} \rightarrow Cl^{37}$ (reference 7).

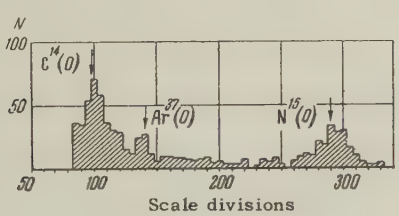

FIG. 4. Proton spectrum from (d, p) reactions on A^{36}, C^{13} and N^{14} at an angle $\theta_{lab.} = 32° 30'$. One scale division = 1.25μ; thickness of the aluminum absorber: 120 mg/cm².

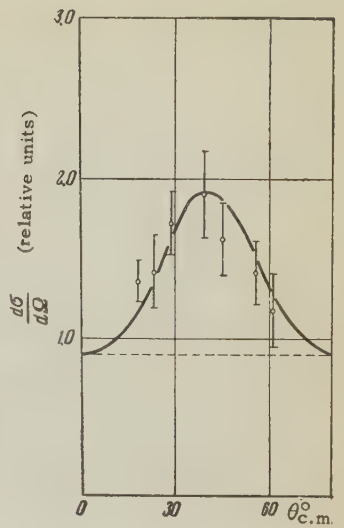

FIG. 5. Angular distribution of protons from the reaction A^{36},(d, p) A^{37} on the ground state, $l_n = 2$.

In conclusion, I should like to express my gratitude to S. S. Vasil'ev for his constant interest in this work. I should also like to thank the cyclotron crew and their director, engineer G. V. Koshelyaev.

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THE TRANSFORMATION $K_2^0 \rightarrow K_1^0$ BY ELECTRONS

Ya. B. ZEL'DOVICH

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The E0 transformation of K_2^0 to K_1^0 by their interaction with electrons is examined. An estimate is given of the cross section for the process and its angular distribution. The interference of the electron and nuclear interactions in the transformation of K_2^0 to K_1^0 in the unscattered beam is considered.

1. INTRODUCTION

As is well known,[1,2] the neutral mesons K_1^0 and K_2^0 have only approximate masses, and their linear combinations K^0 and $\bar{K}^0$, which have definite strangeness, are not neutral in the true sense: that is, for charge conjugation C (or, more exactly, for the combined transform CP[3]) K^0 transforms into $\bar{K}^0$ and $\bar{K}^0$ into K^0. Therefore, it is possible in principle for the interaction with the electromagnetic field to have opposite signs for K^0 and $\bar{K}^0$. In the representation K_1^0, K_2^0 this interaction does not have any diagonal elements by virtue of the true neutrality of K_1^0 and K_2^0. However, such an interaction can have nondiagonal elements capable of producing the transformation of K_2^0 into K_1^0 and K_1^0 into K_2^0. The question has been considered before under the assumption of a K-meson spin different from zero.[4] However, nowadays the K-meson spin is held to be zero. In such a case, as Feinberg showed,[5] the only K interaction linear in the electromagnetic field is the monopole E0 interaction, corresponding to the spherical-condenser model.

The interaction is proportional to div E, that is, to the charge density ρ at the point where the K meson is. This interaction, examined in the present paper, is a basic kind of interaction for electrons with neutral K's and significantly predominates over the weak (Fermi) interaction of electrons with K's.

By virtue of the small mass difference between K_1^0 and K_2^0, the E0-interaction with the field gives only a negligibly small probability for the spontaneous transformation of K_2^0 into K_1^0 in vacuo with a photon emitted. K^0 and $\bar{K}^0$ interact strongly with nucleons, and the effect considered here is only a small correction in the nucleon case.

2. ESTIMATE OF THE E0-MOMENT AND INTERACTIONS WITH ELECTRONS

For an estimate of the E0 moment, equal to $\Sigma\, e_i r_i^2$, we look at a model in which the K^0 meson is represented by the system $(K^+ + \pi^- + \pi^0)$, where K^+ is located at the center, and the π^- is spread over a sphere of radius $\hbar/m_\pi c$.

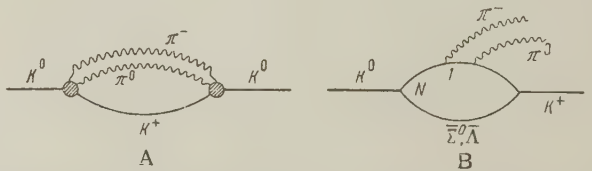
The K^0 transformation in such a system satisfies the selection rules for strong interactions, in particular conservation of strangeness, which forbides the transformation K^0 into $K^- + \pi^+ + \pi^0$. The transformation K^0 into $K^+ + \pi^-$ is forbidden by the pseudoscalarity of the π meson in strong interactions. Other possible virtual states, in particular those considered by Feinberg, require greater expenditures of energy.

Evidently, the estimate obtained, according to which

$$\sum e_i r_i^2 = -e\,(\hbar / m_\pi c)^2,$$

represents an upper limit, and a most likely excessive one.

In Feynman diagram language we consider a type A diagram (see figure) together with a photon which interacts with one or the other of the charged particles. If there is no direct $K-\pi$ interaction, then the hatched circles in A are taken to be Type B diagrams. It is essential that the sign of the contribution of the diagram considered (see figure) does not



depend on the Σ-nucleon mass difference or the difference in their interactions with π's.

However, for the estimate we do not use per-

turbation-theory calculations, but look instead directly at the picture in coordinate space.

Let us find the K-electron interaction energy.* In the field of a potential ϕ, the energy of a spherical condenser is

$$E' = \frac{1}{6}\Delta\varphi \cdot \sum e_i r_i^2.$$

Substituting $\Delta\phi = -4\pi\rho$, we find

$$E' = (2\pi/3)\rho e(\hbar/m_\pi c)^2.$$

The charge density owing to the electron is $\rho = -e|\psi_e|^2$, where e is the absolute value of the elementary charge.

In the center-of-mass system of the colliding K and e, the wavelength of the electron is of the order of $\hbar/m_e c$ or more, that is, significantly larger than the dimensions of the condenser. Therefore, the interaction can be taken to be a point one, with the potential

$$U(\mathbf{r}_K - \mathbf{r}_e) = -\delta(\mathbf{r}_K - \mathbf{r}_e)(2\pi/3)(e^2/\hbar c)(\hbar/m_\pi c)^3 m_\pi c^2 = -B\delta(\mathbf{r}_K - \mathbf{r}_e).$$

The coefficient B of the delta function has dimensions $cm^3 \times$ erg.

3. K SCATTERING ON THE ELECTRON

The interaction described satisfies all the conditions required for the Born approximation to be applicable. Inasmuch as we are only getting estimates, we limit ourselves to the nonrelativistic approximation, and only S waves are taken to be excited.

The solution has the form

$$e^{i\mathbf{p}_0\mathbf{r}} - \frac{m_e}{2\pi\hbar^2}\frac{e^{ip_b r}}{r}\int UdV = e^{i\mathbf{p}_0\mathbf{r}} + \frac{m_e B}{2\pi\hbar^2}\frac{e^{ir}}{r},$$
$$\mathbf{r} = \mathbf{r}_K - \mathbf{r}_e, \qquad m = m_K m_e/(m_K + m_e) \approx m_e.$$

The scattering cross section is equal to

$$\sigma = m_e^2 B^2/\pi\hbar^4 = (4\pi/9)(e^2/\hbar c)^2 (m_e/m_\pi)^2 (\hbar/m_\pi c)^2 \approx 2\cdot 10^{-35}\ cm^2.$$

Feinberg[5] estimates a possible cross section, substituting in his formula a quantity of the dimensions of the pion Compton wavelength. His last formula, in the nonrelativistic limit ($E_e = m_e c^2$) and after multiplying by h and c where necessary, differs from ours only by the factor $9\pi^2/8$. Therefore, the numerical value $10^{-30}\,cm^2$, given in reference 5, is evidently a misprint and should read $10^{-36}\,cm^2$.

The expression derived above for the wave function and the cross section, equal in our work to $2 \times 10^{-35}\,cm^2$, pertain to the K meson. For the energy of the interaction of a $\bar{K}$ with an electron, the constant and the amplitude of the scattered wave have the opposite sign, but the quantity σ is the same as for K.

For the scattering of K_2, which is represented as a linear combination of K and $\bar{K}$, the ratio of the phases of K and $\bar{K}$ in the scattered wave changes sign relative to that in the incident wave. Therefore, if the incident wave is a current of long-lived K_2, the wave scattered by the electron becomes a pure K_1 current (short-lived, decaying into two pions) without any K_2 admixture. The reaction cross section or, more accurately, its upper limit is given in the formula derived above.

We note that the cross section depends neither on the energy (mass) difference between K_2 and K_1 nor on the kinetic energy of the K in the nonrelativistic approximation. The cross section for the transformation of K_2 into K_1 by collision with nucleons is the order of a few millibarns,

In real substances, there is one electron for about every two nucleons. We shall take it that the nuclear cross section for the process $K_2 \rightarrow K_1$, per electron, is on the order of $5 \times 10^{-27} cm^2$. This quantity is 2.5×10^8 times greater than the cross section for the process examined, for electron interactions.

In the laboratory system, the K interactions with nucleons and with electrons have essentially different angular distributions. For K energies of the order of 100 Mev the wavelength is on the order of 1.5×10^{-13}, and the forward differential scattering cross section for nuclei is not more than a few times greater than the average. For the order of magnitude for one electron

$$d\sigma_{nuc}/d\Omega|_{\theta=0} \sim 2\cdot 10^{-28}\ cm^2/\text{steradian}.$$

For scattering by electrons the average energy transfer from 100 Mev K mesons is on the order of 50 kev; almost all the electrons in the atom can be regarded as being free. The spherically symmetric scattering of the center of mass system becomes very strongly peaked in the forward direction on going over to the laboratory system, the maximum angle of inclination of the K meson (in the nonrelativistic approximation) being equal to $\theta_m = m_e/m_K = 10^{-3}$. From this, the differential cross section for forward scattering with no energy loss

$$d\sigma_{el}/d\Omega|_{\theta=0} = (m_K/m_e)^2\sigma_{el}/4\pi = 1.5\cdot 10^{-30}.$$

To this must be added the same cross section for the particles which have transferred the maximum energy (the fraction $4m_e/m_K$) to the electrons. Consequently, the forward cross section for scat-

*From now on we omit the index 0 everywhere, since we only consider neutral K mesons.

tering by electrons, taking the most favorable estimate, is a hundred times less than that for nuclei. When the angle is increased from zero to the maximum inclination θ_m, the cross section increases and the average differential cross section in the interval $0 < \theta < \theta_m$ is four times greater than the one given above.

We note that in the exact forward direction there is a stream of K_1 mesons appearing in the unscattered beam according to the mechanism observed by Case[6] and especially Good,[7] which sharply hinders the observation of the K_1's obtained in the scattering from electrons. From another side, this mechanism leads to effects linear in the scattering amplitude from electrons because of interference with the nuclear interaction. These questions will be considered in detail in the next section.

The remark of Feinberg[5] about the possibility of detecting the electromagnetic scattering of a K^0 by the deviation from charge independence in the scattering of K^0's by protons and neutrons is founded on a misapprehension. The fact is that the K meson has isotopic spin 1/2, K^0 has the isotopic spin projection $-1/2$, and therefore the scattering of K^0 by a proton ($t_z = +1/2$) and on a neutron ($t_z = -1/2$) should not be unique in the theory of the isotopic spin invariance of interactions. The isotopic spin invariance theory establishes connections only between K^0 scattering by a proton and K^+ scattering by a neutron, but for electromagnetic effects the interactions of the K^+ charge with the magnetic moment of the neutron is greater than the E0-interaction of K^0 with the proton. Obviously the experiments are carried out not with free neutrons, but with nuclei, and the K^+ interaction with the charge of the nucleus also enters.

4. CREATION OF K_1 MESONS IN THE UNSCATTERED BEAM

In the very elegant paper of Good[7] there is an exhaustive treatment of the question of the appearance of K_1's in the unscattered beam, a process depending on A(0), the scattering amplitude for zero angle, that is, on the quantity determining the refractive index of the medium n, given by

$$n = 1 + 2\pi N k^{-2} A(0),\quad n' = 1 + 2\pi N k^{-2} A'(0),$$
$$w_1(0) = w_2 k^2 |n - n'|^2 x^2 \delta(\theta)$$
$$= w_2 |A(0) - A'(0)|^2 (\pi N x / k)^2 \delta(\theta).$$

In these formulas n and A are related to K, n' and A' are related to $\bar{K}$, N is the number density of the scattering nuclei, k is the wave vector (inverse wavelength) of the mesons, w_2 is the incident K_2 current, and $w_1(0)$ is the K_1 current formed in the undeflected beam, that is, having precisely the same direction as the primary K_2 beam (the factor $\delta(\theta)$ on the right side of the formula). The peculiarity of the formula is that w_1 is proportional to the square of the thickness of the scatterer.

This peculiarity is connected with the fact that K_1 and K_2 are two very close states whose creation takes place only through weak interactions. The quadratic growth of $w_1(0)$ continues only as long as the time of flight for the path x is less than the time corresponding to a certain mass difference, $\hbar/(m_1 - m_2)c^2$. This time is of the order of magnitude of the decay lifetime of K_1, 10^{-10} sec, which gives $x < 1$ cm for K's with energies of 100 Mev.

The K_1 current, got by means of the elastic scattering into the solid angle $d\Omega$ at an angle θ to the direction of the primary beam K_2, gives the expression

$$dw_1 = {}^1/_4 w_2 |A(\theta) - A'(\theta)|^2 N x d\Omega.$$

In such a way Good establishes the relation between $w_1(0)$ in the direct current and $dw_1/d\Omega$ in the elastically scattered current:

$$w_1(0) = 4\pi^2 N x k^{-2} (dw_1/d\Omega)_{\theta\to 0}\delta(\theta).$$

This relationship is general and for small x is fulfilled for arbitrary (electronic, nuclear) mechanisms of the process $K_2 \to K_1$.

The requirements for the electron process are confined to the fact that the only contribution to $dw_1/d\Omega$ is for small θ, $< 10^{-3}$. Therefore, if $dw_1/d\Omega$ is experimentally determined for angles, say, of $\theta = 0.05$, 0.02, and 0.01, extrapolation to zero does not disclose the contribution from the electron process. Besides this, we have to do experimentally with scatterings on two different kinds of particles, nuclei and electrons; so one of the conditions of Good's results is not met.

Let us take a fresh look at the question and compute directly the contribution to A(0), A'(0) and n, n' from the interaction with the electrons. In Sec. 3 the scattering amplitude of K and $\bar{K}$ in the center of mass system was found. The amplitude of the wave function of the K_1 formed in the collision of a K_2 with an electron is equal to the same quantity, $m_e B/2\pi\hbar^2$. For the transition from the center of mass system to the laboratory system, the forward differential cross section, proportional to the square of the amplitude, is multiplied by the factor $(m_K/m_e)^2$. Consequently, the forward amplitude in the laboratory system is

$$A(0) = -A'(0) = \frac{m_K}{m_e}\frac{m_e B}{2\pi\hbar^2} = \frac{1}{3}\frac{e^2}{\hbar c}\frac{m_K}{m_\pi}\frac{\hbar}{m_\pi c}.$$

From this

$$(n-n')_{el} = 2\pi N_{el} k^{-2} (A - A')$$

$$= m_K \cdot 2BN_{el} / \hbar^2 k^2 = BN_{el} / E_K.$$

The part of the refractive index depending on the interaction between the K and the electron can be easily found directly, without examining the scattering problem. The potential energy of the interaction with an individual electron is $B\delta \times (r_e - r_K)$, which means that the average potential energy of a K in a medium with an electron density N_{el} is $U = N_{el} B$. For the transition of a K from the vacuum into the medium, its kinetic energy changes as a function of the electrons by the ratio $1 + \bar{U}/E_K$, while the wave vector changes by the ratio $\sqrt{1 - \bar{U}/E_K} \approx 1 + \bar{U}/2E_K$, which corresponds to a refractive index

$$n = 1 + N_{el} B / 2E_K, \qquad n' = 1 - N_{el} B / 2E_K$$

in complete correspondence with the formula written above. For iron (density 8 g/cm^3) and 100 Mev K mesons we get numerically $(n-n')_{el} = 1.4 \times 10^{-16}$. Now we find the value of $(n-n')_{nuc}$ corresponding to the chosen nuclear differential cross section 2×10^{-27} cm^2/steradian-nucleon, i.e., for iron, 10^{-25} cm^2/steradian-nucleus. We get

$$|A(0) - A'(0)|^2 = 4 d\sigma / d\Omega = 4 \cdot 10^{-25},$$

$$n - n'|_{nuc} = 2\pi N_{nuc} k^{-2} |A(0) - A'(0)|$$

$$= \pi \left(\frac{\hbar}{m_\pi c}\right)^2 \frac{m_K c^2}{E_K} N_{nuc} |A(0) - A'(0)| = 1.2 \cdot 10^{-15}.$$

The electronic part of the refractive index is real. The relation between the real and imaginary parts of the nuclear difference $|n - n'|_{nuc}$ is unknown. In the most favorable case of real $(n - n')_{nuc}$, the electronic correction can reach ± 25% of the observed number of K_1's in the unscattered beam (proportional to $|(n-n')_{el} + (n-n')_{el}|^2$).

Thus, from Good's general statement the examination of the connection he found between the creation of K_1's at small angles (but always larger than θ_m) and the creation of K_1's in the unscattered beam can give new interesting information about neutral K mesons; the actual divergences between Good's formula and experiment would be shown in the interaction of K's and electrons.

We must remember here that the numerical estimates were made with assumptions that lead to excessive values and that in reality the possibility cannot be excluded that the E0 moment of the K is several times smaller than the chosen value. The effect is lessened in direct proportion.

We shall clear up a possible misunderstanding here. The quantity $(n-n')_{el}$ depends on the density of negative charge from electrons in the substance, but on the average the substance is electrically neutral. Should we not include the electromagnetic interaction of the K with the protons, which are of opposite sign and will compensate for the electron interaction? Actually, in the formation of K_1's on nuclei, the influence of the electromagnetic field of the protons is already included. Because of the local character of the E0 interaction, it acts at all angles, and not at just very small angles. (Above, in the electron case, the small angles arose because of the smallness of the electron mass.)

Consequently, if there were an electromagnetic interaction between the K's and the protons, but not the electrons, Good's formula would be exact. Deviations from it come only because of the interactions with electrons, since $(n-n')_{nuc}$ is taken from experiment.

The experimental investigation of the question poses great difficulties and will be possible only with K_2^0 sources of high intensity and with great accuracy in determining the direction of flight of the K_1, evidently by adding the momenta in the π^+ and π^- decays.

I want to take this opportunity to thank L. D. Landau and L. B. Okun' for their comments.

RESULTS

An estimate of the possible cross section for the process $K_2 + e = K_1 + e$. The ratio of this cross section to the nuclear one is of the order of 10^{-8}, and therefore direct observation of the process is excluded practically. The ratio increases to 1/100 for K_1's traveling in the direction of K_2, that is, forward.

The contribution from the E0-interaction of K_2 with electrons to the refractive index of a substance for the "high" estimate reaches 10%, and so gives a correction of up to 25% in the described relation of Good between the creation of K_1's for small angles and in the unscattered beam. The observation of this correction, although difficult, cannot be excluded.

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ON THE PHASE FACTORS FOR TRANSITION FROM "PARTICLE" TO "HOLE" STATES IN THE NUCLEAR-SHELL THEORY

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The method of second quantization, applied in reference 1 to the calculation of matrix elements for F and G operators of shell theory, is used to obtain a relation between fractional parentage coefficients of the beginning and the end of the shell. The change of the phase factors in the transition from "particle" to "hole" states is investigated. Selection rules for electromagnetic transitions in nuclei due to the symplectic group are found for the case of jj-coupling.

1. In a paper by Tumanov, Shirokov and the author[1] a method based on second quantization was developed to calculate matrix elements of a single particle (F) and a two particle (G) operator for the general case of mixed particle or hole configurations. In the present work we shall obtain some formulae of matrix elements of F operators for hole configurations which have not been given in reference 1. We shall utilize these to obtain a connection between the fractional parentage coefficients for the beginning and the end of a shell, and to obtain new selection rules for electromagnetic transitions in nuclei.

The wave function of a closed shell nucleon state forms a representation of the group of rotations in the usual and the isotopic spin space with $J = 0$ and $T = 0$ and of the symplectic group with $(\sigma) = (00)$. We therefore can consider this state to form the "vacuum" of particles and a state in which one particle is missing from the closed shell as a "hole" in the "vacuum" state. The transition from the nucleon annihilation operator $b(j, m, \tau)$ for the state j, m, τ to the creation operator for the corresponding hole state which has the proper exchange symmetries is given by

$$b(j,\ m,\ \tau) = (-1)^{j+m}(-1)^{1/2+\tau}C^{+}(j,\ -m,\ -\tau), \quad (1)$$

where j is the angular momentum of a nucleon m is its projection and τ is the projection of the isotopic spin.

Utilizing Eq. (36) of reference 1 for the F operator and for the transition of the particle wave function $|\, j^{n_0-n}\ (\sigma)\ JT\rangle$ to the hole wave function $|\, j^{-n}\ (\sigma)\ JT\rangle$ (where $n_0 = 2\ (2j + 1)$ is the number of particles in the closed shell) we express the matrix elements in terms of the reduced matrix elements $\langle j_1 \| f_{k\kappa} \| j_2 \rangle$ after some not too complicated transformations involving Clebsch-Gordan coefficients:

$$\begin{aligned}&\langle j^{-n}(\sigma)\, JT \,|\, f_{k\varkappa} \,|\, j^{-n}(\sigma')\, J'T' \rangle \\ &\quad = -n\,(-1)^{\varkappa}(-1)^{\varkappa'}(-1)^{k}(-1)^{\varkappa} \\ &\quad \times \sum \langle j^{n}(\sigma)\, JT \,|\, j^{n-1}(\widetilde{\sigma})\, \widetilde{J}\widetilde{T}\rangle \langle j^{n}(\sigma')\, J'T' \,|\, j^{n-1}(\widetilde{\sigma})\, \widetilde{J}\widetilde{T}\rangle \\ &\quad \times U\,(\widetilde{J}jJ'k : Jj)\, U\,(\widetilde{T}^{1}/_{2}T'\varkappa : T^{1}/_{2})\, \langle j \| f_{k\varkappa} \| j \rangle. \end{aligned} \quad (2)$$

Here k and κ are the degrees of the operator F in space and isotopic spin space respectively, $U\,(j_1 j_2 J j_3 : J_{12} J_{23})$ are Racah coefficients,[2] $(-1)^{\varkappa}$ and $(-1)^{\varkappa'}$ are phase factors due to the particle $\rightarrow$ hole transition, and $\langle j^{n}\,(\sigma)\ JT |\, j^{n-1}(\sigma')\ J'T' \rangle$ are the fractional parentage coefficients.[3]

In a similar fashion we obtain an expression for mixed hole configurations employing the transformation Eq. (26) of reference 1:

$$\begin{aligned}&\langle j_1^{-(n+1)} J_1 T_1,\ j_2 : JT \,|\, f_{k\varkappa} \,|\, j_1^{-n} J'T' \rangle \\ &\quad = -(-1)^{n}\sqrt{n+1}\,(-1)^{k-j_1-j_2}(-1)^{\varkappa-1}(-1)^{\varkappa}(-1)^{\varkappa'} \\ &\quad \times \left[\frac{2\,(2j_2+1)}{(2k+1)\,(2\varkappa+1)}\right]^{1/2} \langle j_1^{n+1} J_1 T_1 \,|\, j_1^{n} J'T' \rangle\, U\,(J' j_1 J j_2 : J_1 k) \\ &\quad \times U\,(T'^{1}/_{2} T^{1}/_{2} : T_1 \varkappa)\, \langle j_1 \| f_{k\varkappa} \| j_2 \rangle. \end{aligned} \quad (3)$$

2. The matrix element (3), calculated here through the use of the hole concept, can also be obtained in a different fashion, namely by using fractional parentage coefficients for configurations filling the shell by more than one half:

$$\begin{aligned}&\langle j_1^{n_0-(n+1)} J_1 T_1,\ j_2 : JT \,|\, f_{k\varkappa} \,|\, j_1^{n_0-n} J'T' \rangle \\ &\quad = \sqrt{n_0 - n}\,(-1)^{n_0-n-1} \langle j_1^{n_0-n} J'T' \,|\, j_1^{n_0-n-1} J_1 T_1 \rangle \\ &\quad \times U\,(J_1 j_1 J k : J' j_2)\, U\,(T_1{}^{1}/_{2}\, T k : T'^{1}/_{2})\, \langle j_1 \| f_{k\varkappa} \| j_2 \rangle. \end{aligned} \quad (4)$$

Utilizing the known property of the Racah coefficients

$$U(j_1 j_2 J j_3 : J_{12} J_{23})$$
$$= (-1)^{J_{12}+J_{23}-j_1-j_3}\left[\frac{(2J_{12}+1)(2J_{23}+1)}{(2j_1+1)(2j_3+1)}\right]^{1/2} U(J_{12} j_2 J J_{23} : j_1 j_3),$$

we obtain by equating (3) and (4) a relation between the fractional parentage coefficients for the beginning and the end of the shell:

$$\langle j^{n_0-n}(\sigma) JT \mid j^{n_0-(n+1)}(\sigma') J'T' \rangle$$
$$= (-1)^x (-1)^{x'} (-1)^{J-J'-j} (-1)^{T-T'-1/2}$$
$$\times \left[\frac{(n+1)(2J'+1)(2T'+1)}{(n_0-n)(2J+1)(2T+1)}\right]^{1/2}$$
$$\times \langle j^{n+1}(\sigma') J'T' \mid j^n(\sigma) JT \rangle. \qquad (5)$$

3. Setting the phase factors $(-1)^x$ and $(-1)^{x'}$ equal to unity in (5) we arrive at the known result connecting the fractional parentage coefficients for $n > n_0/2$ with those for $n < n_0/2$, derived by Rosenzweig[4] for the case of L-S coupling and extended to j-j coupling by Smirnov.[5] However, such an arbitrary choice of the phases is possible only if we consider the beginning and the end of the shell separately. In going through the middle of the shell it does not work. The formulae obtained in references 4 and 5 do not allow to join the results obtained for $n = n_0/2$ with holes or with particles, and they lead to wrong results when trying to compute double fractional parentage coefficients (for splitting off two particles).

Let us consider as an example the coefficients $\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (\sigma)\ JT \rangle$. Disregarding the phase factors in (5) we obtain

$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (11)\ 20 \rangle = -1/2\sqrt{5},$$
$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (22)\ 20 \rangle = -\sqrt{7}/2\sqrt{5},$$
$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (20)\ 11 \rangle = -3/2\sqrt{5},$$
$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (11)\ 21 \rangle = \sqrt{3}/2\sqrt{5}. \qquad (6)$$

The fractional parentage coefficients must obey the following orthogonality relations:

$$\sum \langle j^n(\sigma) JT \mid j^{n-1}(\sigma') J'T' \rangle \langle j^{n-1}(\sigma') J'T' \mid j^{n-2}(\sigma'') J''T'' \rangle$$
$$\times U(J'' j J j : J' J_0)\, U(T''\ ^1/_2\ T\ ^1/_2 : T' T_0) = 0, \qquad (7)$$

if $J_0 + T_0$ is even. One can easily see that this relation is not fulfilled by the coefficients (6). A direct computation of these coefficients from solving a system of linear equations (the Racah method) yields

$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (11)\ 20 \rangle = 1/2\sqrt{5},$$
$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (22)\ 20 \rangle = -\sqrt{7}/2\sqrt{5},$$
$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (20)\ 11 \rangle = 3/2\sqrt{5},$$
$$\langle (^3/_2)^5 (21)\ ^1/_2\ ^1/_2 \mid (^3/_2)^4 (11)\ 21 \rangle = \sqrt{3}/2\sqrt{5}. \qquad (8)$$

This example illustrates an important circumstance. The first two coefficients of which one changes sign while the other does not correspond to a transition into states with the same J and T but with different (σ). This indicates that the change in the phase in the transition from $n > n_0/2$ to $n < n_0/2$ is connected with the character of the state with respect to the symplectic group. These characteristics were usually disregarded when considering the connection between particle and hole states. They also are not evident in (1). The introduction of additional phase factors into (1) is connected with the peculiarities in the vector addition for hole configurations and has nothing to do with the change of the symplectic character in the transition from particle to hole states. These have to be accounted for separately.

The question on the phase changes in the transition from particle to hole states in the case of LS-coupling has been fully treated by Racah in his original papers on the theory of atomic configurations.[2,6] Considering consecutively the possible transitions in seniority: $v \to v+1$ and $v \to v-1$ he obtains a simple expression connecting the phase change of the configuration $\mid l^n v\, SL \rangle$ with n and v. In connection with nuclei this question, also in LS-coupling, has been investigated by Jahn.[7] The question of the phase change has not been treated for jj-coupling.

According to the group-theoretical classification of states in the scheme of jj-coupling,[8] the wave function for a nucleon state forms a representation of the permutation group, the symplectic group, and the group of three dimensional rotations and consequently can be characterized by the quantum numbers T, (σ) (or, equivalently by s – the seniority and t – the reduced isotopic spin) and J. The problem consists in connecting the phase change in the particle–hole transition with these quantum numbers.

This problem can be solved in the same way as Racah proceeded in the case of atoms.[6] By utilizing the Kasimir operator,[3,8,9] the eigenvalues of which denumerate the representations of the symplectic group, one thus has to obtain a recursion relation for the fractional parentage coefficients

$$\langle j^n(\sigma) JT \mid j^{n-1}(\sigma') J'T' \rangle \text{ and } \langle j^{n-2}(\sigma) JT \mid j^{n-3}(\sigma') J'T' \rangle.$$

This way the transition from particles to holes will not occur at $n = n_0/2$ but at a smaller n. There the choice of the phases is unimportant and one can utilize (5) with an arbitrary choice of the phases. One can easily understand that such a procedure will lead to phases which in the general case will depend linearly on the quantum numbers of the state. The practical difficulty associated with such an approach is due to the fact that here in contrast to the atomic case the seniority is characterized by

two instead of one number. Consequently the number of the possible transitions is much larger. They have to be considered in order to find the dependence of the phase on these quantum numbers.

There exists another simpler procedure for solving this problem. It is based on the fact that the condition of the linear dependence of the phase on s and t allows to construct uniquely general expressions for the phase in terms of these quantum numbers. Since the phase change which is connected with the symplectic group does not depend on J we have

$$(-1)^x = (-1)^{\alpha(n-s)/2+\beta(T-t)}. \qquad (9)$$

Here α and β are certain still undetermined integers (obviously, either zero or unity). We remark that in the atomic case the expression $(-1)^{(n-v)/2}$ is a generalization of Racah's Eq. (65) of reference 6 for the phase. We shall find the constants α and β if we succeed to determine the phases independently of (5) for some arbitrary "basis" set of states. We utilize for this purpose the fact that the orthogonality relations (7) are true for arbitrary n and thus the computation of the fractional parentage coefficients from a set of linear equations (the orthogonality relations) does not only yield their values but also their phases. The "basis" set of states used to obtain α and β can be arbitrary. It just has to be sufficiently large, i.e., it must contain a sufficiently large number of independent equations. This condition is, for example, fulfilled by the above considered set of states $|(^3/_2)^4(\sigma)\, JT\rangle$, which is connected by the fractional parentage coefficients (8) with the state $|(^3/_2)^5\,(21)\,{}^1/_2\,{}^1/_2\rangle$. Comparing (6) and (8) we find $\alpha = 0$, $\beta = 1$. Thus

$$(-1)^x = (-1)^{T-t}. \qquad (10)$$

The constants α and β of (9) do not depend on j. Therefore the relation (10) which has been obtained for $j = 3/2$ has general validity for arbitrary j.

Taking (10) into account, Eq. (5) assumes the final form:

$$\langle j^{n_0-n}(\sigma)\, JT \mid j^{n_0-(n+1)}(\sigma')\, J'T'\rangle = (-1)^{J-J'-j}(-1)^{t-t'-1/2}\left[\frac{(n+1)(2J'+1)(2T'+1)}{(n_0-n)(2J+1)(2T+1)}\right]^{1/2} \times \langle j^{n+1}(\sigma')\, J'T' j \mid j^n(\sigma) JT)\rangle. \qquad (11)$$

In the Appendix we give tables of fractional parentage coefficients which supplement the well known tables of Edmunds and Flowers.[3]

4. We shall now apply the obtained results to the calculation of the probability of electromagnetic transitions in nuclei which consist of half filled shells: $n = n_0/2 = 2j+1$. The matrix element $M_{12}(l, \kappa)$ for the transition of multipolarity l and isotopic multiplicity κ is given by the equation

$$M_{12}(l,\kappa) = \langle j^n(\sigma)\, JT \mid Q(l,\kappa) \mid j^n(\sigma')\, J'T'\rangle = n\Sigma \langle j^n(\sigma)\, JT \mid j^{n-1}(\widetilde{\sigma})\, \widetilde{J}\widetilde{T}\rangle \langle j^n(\sigma')\, J'T' \mid j^{n-1}(\widetilde{\sigma})\, \widetilde{J}\widetilde{T}\rangle \times U(\widetilde{J}jJ'l : Jj)\, U(\widetilde{T}\,{}^1/_2\, T'\kappa : T\,{}^1/_2)\, \langle j \| Q(l,\kappa) \| j\rangle. \qquad (12)$$

The same element can be written down by means of (2). This way we obtain

$$M_{12}(l,\kappa) = -(-1)^l(-1)^\kappa \times (-1)^{T_1-t_1}(-1)^{T_2-t_2} M_{12}(l,\kappa). \qquad (13)$$

Equation (13) expresses a selection rule due to the symplectic group for electromagnetic transitions in the case of jj-coupling. It is not contained in the selection rules with respect to the seniority s which have been given by Neudachin.[10] In particular, if the isotopic spin T_2 and the reduced isotopic spin t_2 in the final state equal zero (e.g., in the ground state of the nucleus) then t_1 and l are connected by the relation

$$t_1 + l = \text{odd number} \qquad (14)$$

In this connection, for example, in Be^8 the E2 transition from the level 20 (22) to the ground state is forbidden while the M1 transition from the level 20 (11) is allowed. Similarly the M1 transitions from the levels 11 (20) and 31 (20) to the level 20 (11) are forbidden as well as E2 transitions from these levels to the level 20 (22). These transitions are allowed by the selection rules of reference 10. Naturally, the rigorous forbiddeness of the given examples will be spoiled by the admixture of the $p_{1/2}$ configuration. However, the indicated regularities can be utilized to identify the levels of this nuclide. Of much greater interest is the application of this selection rule to heavier nuclei where jj-coupling obtains. Selection rules, analogous to (13), can be also obtained for LS-coupling.

5. In conclusion we give the formula for the particle to hole transition of double fractional parentage coefficients which follows immediately from Eqs. (32) and (48) of reference 1:

$$\langle j^{n_0-n}(\sigma)\, JT \mid j^{n_0-(n+2)}(\sigma')\, J'T',\ j^2 J_0 T_0\rangle = (-1)^{J-J'}(-1)^{t-t'} \times \left[\frac{(n+2)(n+1)}{(n_0-n)(n_0-n-1)}\frac{(2J'+1)(2T'+1)}{(2J+1)(2T+1)}\right]^{1/2} \times \langle j^{n+2}(\sigma')\, J'T' \mid j^n(\sigma)\, JT,\ j^2 J_0 T_0\rangle.$$

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APPENDIX

Fractional parentage coefficients $\langle (^3/_2)^5\, JT\ (\sigma) | (^3/_2)^4 (\sigma')\, J'T' \rangle$

$J'T'$ (σ')	JT (σ)				
	$^1/_2\ ^1/_2$ (21)	$^3/_2\ ^1/_2$ (10)	$^5/_2\ ^1/_2$ (21)	$^7/_2\ ^1/_2$ (21)	$^3/_2\ ^3/_2$ (10)
00 (00)	—	$1/\sqrt{10}$	—	—	—
20 (11)	$1/2\sqrt{5}$	$-\sqrt{3/10}$	$-1/2\sqrt{5}$	$-1/2\sqrt{5}$	—
20 (22)	$-\sqrt{7}/2\sqrt{5}$	—	$-\sqrt{5}/2\sqrt{7}$	$1/\sqrt{35}$	—
40 (22)	—	—	$-\sqrt{6/35}$	$-3/2\sqrt{7}$	—
11 (20)	$3/2\sqrt{5}$	$-\sqrt{3}/5\sqrt{2}$	$\sqrt{21}/10$	—	$-\sqrt{3}/2\sqrt{5}$
21 (11)	$\sqrt{3}/2\sqrt{5}$	$\sqrt{2/5}$	$-\sqrt{3}/2\sqrt{5}$	$-\sqrt{3}/2\sqrt{5}$	$-^1/_2$
31 (20)	—	$-\sqrt{7}/5\sqrt{2}$	$-\sqrt{6}/5$	$3/2\sqrt{5}$	$-\sqrt{7}/2\sqrt{5}$
02 (00)	—	—	—	—	$^1/_2$

Fractional parentage coefficients $\langle (^3/_2)^6\, JT\ (\sigma) | (^3/_2)^5 J'T'\ (\sigma') \rangle$

$J'T'$ (σ')	JT (σ)			
	01 (00)	10 (20)	21 (11)	30 (20)
$^1/_2\ ^1/_2$ (21)	—	$1/\sqrt{3}$	$1/\sqrt{15}$	—
$^3/_2\ ^1/_2$ (10)	$-\sqrt{5}/3$	$-1/\sqrt{5}$	$1/3\sqrt{5}$	$1/\sqrt{5}$
$^5/_2\ ^1/_2$ (21)	—	$\sqrt{7/15}$	$-1/\sqrt{5}$	$2\sqrt{2}/\sqrt{35}$
$^7/_2\ ^1/_2$ (21)	—	—	$2/\sqrt{15}$	$2/\sqrt{7}$
$^3/_2\ ^3/_2$ (10)	$^2/_3$	—	$^2/_3$	—

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THE NUCLEAR INTERACTION IN THE SCATTERING OF CHARGED PARTICLES FROM NONSPHERICAL NUCLEI

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The nuclear interaction in the scattering of charged particles with energies close to the height of the Coulomb barrier from black, nonspherical nuclei is considered.

IN a previous paper[1] the author obtained an expression for the wave function of a charged particle scattered from a nonspherical nucleus, containing the nuclear amplitudes $b^{\Omega}_{ll'}$. The aim of the present paper is to actually calculate these amplitudes for the case of a black nucleus.* This case has practical interest, since the condition of complete absorption of the incoming particles in the nucleus is, apparently, well satisfied for α particles (which are widely used in Coulomb excitation experiments) in the energy region under consideration ($E \approx 20$ Mev). The determination of the nuclear amplitudes (23.1) is in general very involved, since the potential, and therefore the wave functions, have quite different symmetries inside and outside the nucleus. The conditions for the joining of the wave function at the nuclear surface therefore lead to a complicated system of algebraic equations for the amplitudes $b^{\Omega}_{ll'}$. It is obvious that this system can be solved only numerically. However, this situation becomes very much simpler in the case of a black nucleus, since it is known[2] that in this case the wave function satisfies on the nuclear surface the condition

$$\frac{\partial}{\partial r}(r\Psi) = -iK(r\Psi), \tag{1}$$

where K is a certain constant. With this boundary condition the amplitudes $b^{\Omega}_{ll'}$ can be found without considering the solution in the internal region at all.

1. BOUNDARY CONDITION FOR A DIFFUSE SURFACE

In writing down the boundary condition (1), where K is the complex wave vector of the particle inside the nucleus, we assumed that the nucleus has a well-defined boundary. In actual fact, the nuclear potential drops smoothly over a distance d which is of the order of magnitude of the range of the nuclear forces. Since $Kd \gtrsim 1$, this circumstance can have an appreciable effect on the scattering amplitude.

We are interested in the region of energies close to the height of the Coulomb barrier. To take account of the diffuseness of the boundary we can use the method proposed by Gribov.[3] He showed that, with an accuracy up to and including terms of order kd (where k is the wave number of the particle outside the nucleus),

$$d^2u/dr^2 = (2m/\hbar^2)Vu, \tag{2}$$

in the diffuse surface region. Here u is the wave function multiplied by r, and V is the nuclear potential (the particle is assumed to be neutral). It is easily seen that the case of a charged particle is different from that considered in reference 3 mainly by the fact that the role of the wave number k is here taken over by the quantity

$$k' = \sqrt{2m|E - V_c|}/\hbar,$$

where V_C is the Coulomb potential. Up to terms of order k'd, the wave function therefore satisfies Eq. (2), as before.

We write the nuclear potential in the diffuse surface region in the form

$$V = \frac{\hbar^2}{2m}K^2 v\left[\frac{R_0(\theta) - r}{d}\right], \tag{3}$$

where v is a dimensionless function with the property

$$v(x) \to 1, \quad x > 1; \tag{4}$$
$$v(x) \to 0, \quad x < -1. \tag{5}$$

At the internal end of the diffuse region we have again the old boundary condition (1). If, therefore, $\varphi_1[(R_0(\theta) - r)/d]$ is a solution of equation (2) of the form

$$\varphi_1 \sim e^{-iKr} \quad \text{for } v \sim 1, \tag{6}$$
$$\varphi_1 \sim a + b[R_0(\theta) - r] \quad \text{for } v \sim 0, \tag{7}$$

where a and b are constants uniquely determined

*We shall use the notations of reference 1. Reference to the formulas of this paper will be made in the form, e.g., (21.1).

by the form of the function v, then the wave function multiplied by r is, in the diffuse region,

$$u = A(\theta)\varphi_1. \tag{8}$$

For values of r corresponding to $v \approx 0$, this function must join smoothly to the wave function, multiplied by r, in the external region, which we denote by u^0. Expanding $u^0(r, \theta)$ in powers of $R_0(\theta) - r$ and discarding terms of order $(k'd)^2$, we obtain

$$u^0 \approx u^0(R_0(\theta), \theta)\left[1 - \frac{1}{u^0(R_0)}\frac{\partial u^0}{\partial r}\Big|_{r=R_0}(R_0(\theta) - r)\right]. \tag{9}$$

Comparing (9) and (7), we find the necessary conditions for equality of the two functions for $v \approx 0$:

$$\partial(r\Psi)/dr = -iK_{\text{eff}}(r\Psi) \quad \text{for } r = R_0(\theta), \tag{10}$$

where

$$K_{\text{eff}} = -ib/a. \tag{11}$$

The (complex) quantity b/a depends on the form of the function v. For example, in the case $v(x) = 1/(1 + e^{-x})$, which is important for applications,

$$K_{\text{eff}} = K \tanh \pi\gamma / \pi\gamma, \tag{12}$$

where $\gamma = Kd$. Hence the diffuseness of the boundary leads to the replacement of K by K_{eff} in the boundary condition (1).

2. THE SYSTEM OF EQUATIONS FOR THE AMPLITUDES $b^{\Omega}_{ll'}$

If the wave function Ψ is written in the form (24.1), the boundary condition (11) must be fulfilled for all functions (23.1).

We introduce the notations

$$dF_{l\Omega}(\rho)/d\rho = f_{l\Omega}(\rho)F_{l\Omega}(\rho),$$

$$dG_{l\Omega}(\rho)/d\rho = g_{l\Omega}(\rho)G_{l\Omega}(\rho), \tag{13}$$

$$\frac{F_{l\Omega}[\rho(\mu)]}{F_{l\Omega}} = \varphi_{l\Omega}(\mu), \qquad \frac{G_{l\Omega}[\rho(\mu)]}{G_{l\Omega}} = \gamma_{l\Omega}(\mu), \tag{14}$$

$$G_{l\Omega}(\rho) = H_{l\Omega}(\rho) - iF_{e\Omega}(\rho), \tag{15}$$

where $\rho(\mu) = kR_0(\mu)$, $F_{l\Omega} = F_{l\Omega}[\rho(1)]$, and $R_0(\mu)$ determines the surface defining the shape of the nucleus according to (3). We use elliptic coordinates [see (14.1)]. If the weak dependence on ρ of the logarithmic derivatives $f_{l\Omega}$ and $g_{l\Omega}$ on the nuclear surface is neglected, the boundary condition (11) for the function leads to the equation

$$\varphi_{l\Omega}(\mu)\Phi_{l\Omega}(\mu) = \sum_{l'}{}' \beta^{\Omega}_{ll'}\gamma_{l'\Omega}(\mu)\Phi_{l'\Omega}(\mu)$$

$$+ i\sum_{l'}{}' \beta^{\Omega}_{ll'}\varphi_{l'\Omega}(\mu)\Phi_{l'\Omega}(\mu), \tag{16}$$

where

$$b^{\Omega}_{ll'} = -\beta^{\Omega}_{ll'}F_{l\Omega}(f_{l\Omega} + iK_{\text{eff}})/G_{l\Omega}(g_{l\Omega} + iK_{\text{eff}}). \tag{17}$$

The coefficients $\beta^{\Omega}_{ll'}$ can now be found by the same method as that used by Gol'din et al.[4] for the approximate solution of the problem of the α decay of a nonspherical nucleus. Thus we multiply (17) by $\Phi^*_{n\Omega}(\mu)/\gamma_{n\Omega}(\mu)$ and integrate over μ. At energies of the incoming particle below the height of the Coulomb barrier B, the ratio $\gamma_{l\Omega}(\mu)/\gamma_{n\Omega}(\mu)$ depends weakly on μ in the quasiclassical approximation (which we are considering here). (The functions $\gamma_{ln}(\mu)$ themselves do, of course, fluctuate strongly over the surface of the nucleus). Therefore all integrals

$$\int_{-1}^{1} \Phi^*_{n\Omega}(\mu)\frac{\gamma_{l\Omega}(\mu)}{\gamma_{n\Omega}(\mu)}\Phi_{l\Omega}(\mu)\,d\mu \tag{18}$$

are small for $n \neq l$ and can be neglected in our approximation (a special estimate shows that this involves an error of order $\beta = \Delta R/R$).

For the $\beta^{\Omega}_{ll'}$ we then obtain the following system of algebraic equations:

$$\beta^{\Omega}_{ln} = I^{\Omega}_{ln} - i\sum_{l'}{}' \beta^{\Omega}_{ll'}I^{\Omega}_{l'n}(f_{l'\Omega} + iK_{\text{eff}})$$

$$\times F_{l'\Omega}/(g_{l'\Omega} + iK_{\text{eff}})G_{l'\Omega}, \tag{19}$$

where

$$I^{\Omega}_{ln} = \int \Phi^*_{n\Omega}(\mu)\frac{\varphi_{l\Omega}(\mu)}{\gamma_{n\Omega}(\mu)}\Phi_{l\Omega}(\mu)d\mu. \tag{20}$$

The terms under the sum on the right hand side of (19) are proportional to the factor $F_{l\Omega}/G_{l\Omega}$, which decreases fast with increasing l (roughly speaking, as $\exp(-l^2/\eta^{4/3})$. In a first approximation we can therefore retain only the first 2 or 3 terms in the sum. The $\beta^{\Omega}_{ll'}$ can then easily be expressed in terms of the integrals I^{Ω}_{ln}. If desired, the discarded terms can be accounted for by a method of successive approximations. In order to calculate I^{Ω}_{ln}, one must know the functions $F_{l\Omega}(\rho)$ and $G_{l\Omega}(\rho)$ near the classical turning point ρ_0. Since we consider this problem in the quasiclassical approximation, we can use the approximate solution of equation (17.1) in terms of the Airy function:[5,6]

$$F_{l\Omega}(\rho) = (t/g)^{1/4}v(-t), \tag{21}$$

$$G_{l\Omega}(\rho) = (t/g)^{1/4}u(-t), \qquad t = \left[\frac{3}{2}\int_{\rho_u}^{\rho} g^{1/2}\,d\rho\right]^{2/3}, \tag{22}$$

$$g(\rho) = 1 - \frac{2\eta\rho}{\rho^2 - c^2} - \frac{\Lambda_{l\Omega} - c^2}{\rho^2 - c^2} - \frac{c^2(\Omega^2 - 1)}{(\rho^2 - c^2)^2}, \qquad (23)$$

where ρ_0 is the value of ρ for which $g(\rho) = 0$. v and u are the Airy functions tabulated by Fock.[6]

We can simplify the integral (23) by making use of the fact that we are only interested in energy values E very close to B, and therefore only in values ρ close to ρ_0. We expand the expression under the root sign in powers of $\rho_0 - \rho$ and retain only the first term. Then we have

$$F_{l\Omega}(\rho) = C^{-1/6} v(-x), \qquad G_{l\Omega}(\rho) = C^{-1/6} u(-x), \qquad (24)$$

where

$$x = C^{1/3}(\rho - \rho_0), \qquad C = dg/d\rho \quad \text{for } \rho = \rho_0.$$

It is easily seen that the next term is of order $\eta[(B-E)/B]^{5/2}$. The above expression is therefore valid if the latter quantity is small compared to unity.

3. LIMITING VALUE OF THE AMPLITUDES $b^{\Omega}_{ll'}$ for E < B

We obtain a particularly simple result, if the energy of the incoming particle is so much smaller than the barrrier height, that the sum on the right hand side of equation (19) can be neglected altogether.

For this we must satisfy the condition

$$\eta[(B-E)/B]^{3/2} > 1, \qquad (25)$$

since the "radial" functions $F_{l\Omega}(\rho)$ and $G_{l\Omega}(\rho)$ depend on the number l in essentially the same way as the corresponding Coulomb functions F_l and G_l, so that

$$\frac{F_{l\Omega}(\rho)}{G_{l\Omega}(\rho)} \leqslant \frac{F_0(\rho)}{G_0(\rho)} \approx \exp\left\{-\frac{8}{3}\eta\left(\frac{B-E}{B}\right)^{3/2}\right\}.$$

The applicability of the adiabatic approximation used by us requires that, together with (25), also

$$(B-E)/B \ll 1. \qquad (26)$$

The integral I^{Ω}_{l} can be computed approximately by using the fact that the function $\varphi_{l\Omega}(\mu)/\gamma_{n\Omega}(\mu)$ has a sharp maximum at $|\mu| = 1$.

Introducing the new variable $\mu = \cos\theta$, we can write, for $\theta \ll 1$:

$$\varphi_{l\Omega}(\cos\theta)/\gamma_{n\Omega}(\cos\theta) \approx e^{-a\theta^2}, \qquad (27)$$

where

$$a = [f_{l\Omega} - g_{n\Omega}]\, d\rho(\theta)/d(\theta^2) \quad \text{for } \theta = 0. \qquad (28)$$

Since $d\rho/d(\theta^2) \approx 2\eta\beta$, the quasi-classical estimate

$$f_{l\Omega} \approx -g_{l\Omega} \approx \left(\frac{B-E}{B}\right)^{1/2},$$

leads to

$$a \approx 4\eta\beta\left(\frac{B-E}{B}\right)^{1/2}.$$

If the conditions (25) and (26) are satisfied, this quantity is practically always large in comparison with unity, and therefore only the values $|\mu|$ close to unity (i.e., θ close to zero) are essential in the integral (20).

Since $\theta \ll 1$, $\Phi_{l\Omega}(\cos\theta) \sim \theta^{\Omega}$, we have $I^{\Omega}_{ln} \sim 1/a^{2\Omega+1}$. Up to terms of order $1/a^2$ we can therefore assume

$$I^{\Omega}_{ln} = 0 \quad \text{for } \Omega \neq 0. \qquad (29)$$

For the evaluation of I^{0}_{ln} one can use the representation (20.1) for the function $\Phi_{l\Omega}$, replacing $Y_{l0}(\theta, \varphi)$ by $\sqrt{(2l+1)/4\pi}\, J_0[(l+\frac{1}{2})\theta]$ and integrating over θ from zero to infinity. We then obtain the following result for the amplitudes $b^{\Omega}_{ll'}$:

$$b^{\Omega}_{ll'} = 0, \qquad \Omega \neq 0, \qquad (30)$$

$$b^{0}_{ll'} = -\frac{(f_{l0} + iK_{\text{eff}})F_{l0}}{(g_{l'0} + iK_{\text{eff}})G_{l'0}} \frac{1}{a} \sum_{n,k} C_{ln} C_{l'k} [(n+1/2)(k+1/2)]^{1/2}$$

$$\times \exp\left[-\frac{(n+1/2)^2 + (k+1/2)^2}{4a}\right] I_0\left[\frac{(n+1/2)(k+1/2)}{2a}\right], \qquad (31)$$

where I_0 is the Bessel function of imaginary argument.

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STABILITY OF AN IDEALLY CONDUCTING LIQUID FLOWING BETWEEN CYLINDERS ROTATING IN A MAGNETIC FIELD

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Sufficient conditions for the stability of an ideal liquid flowing in axial and toroidal magnetic fields are derived. Critical values of the magnetic fields which stabilize the flow are obtained and a physical interpretation of the results is presented.

1. INTRODUCTION

RAYLEIGH,[1] Taylor,[2] Meksyn,[3] and Synge[4] investigated the stability of flow of a viscous, incompressible liquid between rotating cylinders. The classical condition of stability (see Landau and Lifshitz[5])

$$\Omega_1 R_1^2 < \Omega_2 R_2^2 \tag{1.1}$$

follows from the conservation of angular momentum of a liquid particle (Ω_1, Ω_2 are the angular velocities and R_1, R_2 are the radii of the rotating cylinders).

We examine in this paper the stability of flow of an ideally conducting nonviscous liquid in a magnetic field. Chandrasekhar[6] solved a similar problem for a poorly-conducting viscous liquid.

The motion of the liquid and the field in it are described by the equations of magnetic hydrodynamics for an ideally conducting liquid:

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla\Phi + \frac{1}{4\pi}(\mathbf{B}\nabla)\mathbf{B} + \nu\nabla^2\mathbf{V}, \quad \operatorname{div}\mathbf{V} = 0,$$
$$\frac{d\mathbf{B}}{dt} = (\mathbf{B}\nabla)\mathbf{V}, \quad \operatorname{div}\mathbf{B} = 0 \tag{1.2}$$

(V is the velocity, B is the induction of the magnetic field, and Φ is the total pressure of the substance and the field).

Independent of the magnitude of the viscosity, these equations admit of the following stationary solution:

$$\mathbf{V}_0 = (ar + e/r)\,\mathbf{e}_\varphi, \quad \mathbf{B}_0 = \mathbf{B}_{0\varphi}(r)\,\mathbf{e}_\varphi + \mathbf{B}_{0z}\mathbf{e}_z,$$
$$a = -(\Omega_1 R_1^2 - \Omega_2 R_2^2)/(R_2^2 - R_1^2),$$
$$e = (\Omega_1 - \Omega_2) R_1^2 R_2^2/(R_2^2 - R_1^2). \tag{1.3}$$

In the following discussion we ignore the effect of viscosity on perturbations of the stationary flow.

We investigate the "linear" stability of flow, i.e., the stability relative to disturbances of infinitesimal amplitude. The system (1.2), after substituting the disturbed values of velocity, field, and pressure

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{v}, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \Phi = \Phi_0 + \phi, \tag{1.4}$$

neglecting viscosity and linearization, becomes a system of linear differential equations with coefficients that are independent of z, φt in cylindrical coordinates. Solutions of such a system have the form

$$\mathbf{v} = \mathbf{v}(r)\exp\{i(m\varphi + kz + \omega t),$$
$$\mathbf{b} = \mathbf{b}(r)\exp\{i(m\varphi + kz + \omega t)\},$$
$$\phi = \phi(r)\exp\{i(m\varphi + kz + \omega t)\}. \tag{1.5}$$

Only disturbances of definite ω satisfy the homogeneous boundary conditions $v_r = 0$, $b_r = 0$ for $r = R_1$, R_2. The sign of the imaginary part Im $\omega = \omega_1$ determines the stability of flow.

2. STABILITY OF FLOW IN AN AXIAL MAGNETIC FIELD

As can be easily shown, the most dangerous disturbances are the ones that do not depend on φ. The amplitudes of velocity and field disturbances are expressed in terms of the amplitude of the radial component of disturbance of the magnetic field in the following manner

$$v_r = (\omega/k)\,b_r/B_{0z}; \tag{2.1}$$

$$v_\varphi = \frac{2i}{k(\omega^2/k^2 - B_{0z}^2/4\pi\rho)}\left(\frac{\omega^2}{k^2}a + \frac{B_{0z}^2}{4\pi\rho}\frac{e}{r^2}\right)\frac{b_r}{B_{0z}}; \tag{2.2}$$

$$v_z = (i\omega/k^2 B_{0z})(db_r/dr + b_r/r); \tag{2.3}$$

$$b_\varphi = \frac{2i\omega}{k^2}\,\frac{a + e/r^2}{B_{0z}^2/4\pi\rho - \omega^2/k^2}\,b_r; \tag{2.4}$$

$$b_z = (i/k)(db_r/dr + b_r/r), \tag{2.5}$$

while the amplitude b_r itself is found from the equation

$$\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega^2}{k^2}\right)^2\left\{\frac{d}{dr}r\frac{d}{dr}-\frac{1}{r}-rk^2\right\}b_r+4\frac{\omega^2}{k^2}a\left(ar+\frac{e}{r}\right)b_r$$

$$+\frac{B_{0z}^2}{\pi\rho}\frac{e}{r^2}\left(ar+\frac{e}{r}\right)b_r=0 \qquad (2.6)$$

with boundary conditions

$$b_r=0 \quad \text{for } r=R_1, R_2. \qquad (2.7)$$

We shall show that ω^2 is a real number; i.e., there is possible either an aperiodic growth in the disturbances or an oscillation about the equilibrium point. Actually, multiplying Eq. (2.6) by b_r^*, the complex conjugate of b_r, we obtain

$$J=\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega^2}{k^2}\right)^2J_1-4\frac{\omega^2}{k^2}aJ_2-\frac{B_{0z}^2}{\pi\rho}eJ_3=0, \qquad (2.8)$$

where

$$J_1=\int_{R_1}^{R_2}(r|b_r'|^2+\frac{1}{r}|b_r|^2+rk^2|b_r|^2)\,dr>0; \qquad (2.9)$$

$$J_2=\int_{R_1}^{R_2}V_{0\varphi}(r)|b_r|^2\,dr; \qquad (2.10)$$

$$J_3=\int_{R_1}^{R_2}V_{0\varphi}(r)|b_r|^2\frac{dr}{r^2}. \qquad (2.11)$$

The imaginary part of (2.8) is

$$\operatorname{Im}J=\frac{4\omega_i\omega_r}{k^2}\left\{\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega_r^2}{k^2}+\frac{\omega_i^2}{k^2}\right)J_1+2aJ_2\right\}=0. \qquad (2.12)$$

It is easy to see that the expression in braces is not equal to zero, for otherwise the real part of (2.8) would not vanish:

$$\operatorname{Re}J=-\left\{\left[\left(\frac{B_{0z}^2}{4\pi\rho}-\frac{\omega_r^2}{k^2}+\frac{\omega_i^2}{k^2}\right)^2\right.\right.$$

$$\left.\left.+\frac{4\omega_i^2\omega_r^2}{k^4}\right]J_1+\frac{B_{0z}^2}{\pi\rho}J_4\right\}<0, \qquad (2.13)$$

where

$$J_4=\int_{R_1}^{R_2}V_{0\varphi}^2|b_r|^2\frac{dr}{r}>0,$$

which contradicts Eq. (2.8). Therefore $\omega_1\omega_r=0$.

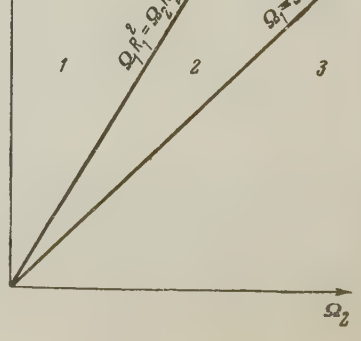
A sufficient condition for stability follows from expression (2.8). If the cylinders are rotating in the same direction, then $J_2>0$ and $J_3>0$, because $V_{0\varphi}>0$. Therefore when $e<0$ (we know that $a>0$) the flow is stable. The sufficient condition for stability has the form:

$$\Omega_2-\Omega_1>0. \qquad (2.14)$$

The same condition can be obtained more obviously by the Rayleigh method. Let us examine an element of liquid rotating in some layer with angular velocity Ω. If the element departs from the layer, then because of the ideal conductivity it will drag away the line of force to which it is "glued." Almost all of the line of force will remain in the original layer. Therefore it rotates with the former angular velocity and drags the fluid element after it. As a result, the element retains its previous angular velocity. But without a field, the axial momentum component is preserved, because there are no tangential forces acting on the element. Formally, the preservation of angular velocity during slow motion of the particle follows directly from expressions (2.1) through (2.5).

The element experiences three radial forces in the new layer: the quasi-elastic force of the distorted magnetic line of force, the centrifugal force, and the equilibrium pressure gradient in the new layer. If $d\Omega^2/dr>0$, i.e., condition (2.14) is not fulfilled, the centrifugal force exceeds the equilibrium pressure gradient. Thus, $\Omega_1>\Omega_2$ is the necessary condition for the existence of the centrifugal effect. In a sufficiently weak field [see (2.8)] this effect leads to instability of flow.

Without a field, similar arguments (see reference 5) lead to the condition for the existence of the centrifugal effect $\Omega_1R_1^2>\Omega_2R_2^2$ (Rayleigh condition). However, when $B\to 0$ condition (2.14) does not transform into the Rayleigh condition (see the figure). A paradox arises upon going in the limit from $B\to 0$ to $B=0$. Actually, at any large but finite conductivity in a weak field, if the disturbance has a long enough development time [see (2.20)], the field has enough time to diffuse from the disturbance and the element no longer retains its angular velocity. In this sense the situation is similar to the well-known transition from a slightly viscous to a nonviscous liquid. For example, flow between flat plates is stable when $R_g=\infty$ and unstable at any large but finite Reynolds number R_g. This is connected with the fact that stability of flow results from the law of conservation of the curl of the velocity. No matter how small the viscosity, the curl of the veloc-



Stability region of the Couette flow with (3) and without (2+3) field.

ity has time to diffuse from the disturbances near the resonance layer (the "layer of internal friction"), where the phase velocity of the disturbances is equal to the velocity of flow. Therefore the mechanism of stabilization does not operate near this layer (see reference 8).

The neglect of viscosity and of ohmic resistance is justified consequently, by the smallness of characteristic times as compared with diffusion times, i.e., by the fact that the parameters

$$P_m = B_{0z}\sigma\sqrt{4\pi}L/c^2\sqrt{\rho}, \tag{2.15}$$

$$P_g = B_{0z}L/\nu\sqrt{4\pi\rho} \tag{2.16}$$

are large as compared to unity (L is the characteristic length).

Neutral disturbances, as seen from expressions (2.1) through (2.5), are barrel-shaped disturbances of the field, brought about by azimuthal current. With this, the liquid tends to rotate as a solid body, while the velocity gradient builds up in the region of the enhanced field.

In a sufficiently strong field the flow is stabilized by a large quasi-elastic force. The critical value of the field can be found easily for the particular case of a small space between the cylinders. If the cylinders are rotating in the same direction, the coefficients in Eq. (2.6) can be replaced by their average values, and the equation can be solved in trigonometric functions; but if the cylinders are rotating in opposite directions, we must account for the linear terms, and Eq. (2.6) will reduce to the Airy equation. In the first case we have for disturbances with wave number k

$$\omega^2 = k^2\left\{\frac{B_{0z}^2}{4\pi\rho} - \frac{2\Omega_0(\Omega_1-\Omega_2)\,dR_0}{\pi^2+k^2d^2}\right\}, \tag{2.17}$$

where, as above, $d = R_2 - R_1$, $R_0 = \frac{1}{2}(R_2 + R_1)$, and $\Omega_0 = \Omega(R_0)$. When $k^2 \to 0$ we have $\omega^2 \to 0$, because long-wave disturbances interact too weakly with the field. When $k^2 \to k^2_{max}$ we have $\omega^2 \to 0$, because the short-wave disturbances distort the lines of force too much. From (2.17) it follows that to suppress the instability it is sufficient that

$$B_{0z}^2/8\pi > \rho\Omega_0(\Omega_1-\Omega_2)R_0d/\pi^2. \tag{2.18}$$

The wave number of dangerous disturbances is bounded from above,

$$k^2 \leqslant k^2_{max} = (\Lambda-\pi^2)/d^2, \tag{2.19}$$

as is the increment

$$\omega_i < \omega_{imax} = (B_{0z}\sqrt{\pi}/2\sqrt{\rho}\,d)(\sqrt{\Lambda}/\pi - 1), \tag{2.20}$$

while

$$\Lambda = 8\pi\rho\,\Omega_0(\Omega_1-\Omega_2)R_0d/B_{0z}^2. \tag{2.21}$$

For cylinders rotating in opposite directions, the results do not change qualitatively.

The flow near a cylinder rotating in an unbounded medium is unstable in a sufficiently weak field, as follows from condition (2.14). However, the same can be shown by a direct calculation. By substituting $b_r = \varphi/\sqrt{r}$ we transform (2.6) into a Schrödinger-type equation

$$\frac{d^2\psi}{dr^2} + \left\{-k^2 - \frac{3}{4}\frac{1}{r^2} + \frac{B_{0r}^2}{\pi\rho}\frac{(\Omega_1R_1)^2}{r^4(B_{0z}^2/4\pi\rho - \omega^2/k^2)^2}\right\}\psi = 0, \tag{2.22}$$

since $e = \Omega_1R_1^2$ and $a = 0$. The boundary conditions have the usual appearance: $\varphi = 0$ when $r = R_1$ and ∞. Therefore, if we have in the well, at $\omega^2 < 0$

$$U = \frac{3}{4}\frac{1}{r^2} + \frac{B_{0z}^2}{\pi\rho}\frac{\Omega_1^2R_1^4}{r^4(B_{0z}^2/4\pi\rho-\omega^2/k^2)^2} \text{ for } r > R_1,$$

$$U = \infty \text{ for } r < R_1,$$

and levels with $E = -k^2 < 0$ exist, the flow is unstable. In the quasi-classical approximation, the condition for the existence of a level has the form

$$\int_{R_1}^{R_2}\sqrt{-k^2-U}\,dr = \pi(n+\tfrac{3}{4}) \tag{2.23}$$

(see reference 9) where n is the number of zeros of the ψ functions in the well. For long waves, $k^2 \ll 1/R_0$, we obtain the condition:

$$\sqrt{R_0^2/R_1^2-1} + \sin^{-1}(R_1/R_0)$$

$$= (2\pi/\sqrt{3})(n+{}^3/_4) + \pi/2, \tag{2.24}$$

where $$R_0 = \frac{2B_{0z}}{\sqrt{3\pi\rho}}\frac{\Omega_1R_1^2}{(B_{0z}^2/4\pi\rho - \omega^2/k^2)}.$$

To satisfy the quasi-classical approximation condition we set $n = 10$. Then

$$-\frac{\omega^2}{k^2} \approx \frac{B_{0z}}{\sqrt{4\pi\rho}}\left\{\frac{\Omega_1R_1}{18} - \frac{B_{0z}}{\sqrt{4\pi\rho}}\right\},$$

and in a field less than $\sqrt{4\pi\rho}\,\Omega_1R_1/18$ the flow is known to be unstable. The critical value of the field is obviously larger, but its determination requires an accurate solution of the problem.

Since the instability has a local character, a simple vortex is also unstable in an axial magnetic field of less than critical value. This instability can appear, for example, when a star rotates in its own magnetic field.

3. STABILITY OF FLOW IN AN AZIMUTHAL MAGNETIC FIELD

In the presence of a magnetic field directed along φ, ω^2 is determined as the eigenvalue of the differential equation for the radial component

of velocity disturbance:

$$v_r'' + \frac{v_r'}{r} - \left\{k^2 + \frac{1}{r^2} - 4a\left(a + \frac{e}{r^2}\right)\frac{k^2}{\omega^2} + \frac{B_{0\varphi}}{2\pi\rho r}\frac{k^2}{\omega^2}\left(\frac{dB_{0\varphi}}{dr} - \frac{B_{0\varphi}}{r}\right)\right\} v_r = 0, \quad (3.1)$$

with boundary conditions

$$v_r = 0 \text{ for } r = R_1, R_2. \quad (3.2)$$

The problem is easily solved for a narrow space between cylinders rotating in the same direction, as in Sec. 2. With this

$$\omega^2 = \left[4a\Omega_0 - \frac{B_{0\varphi}}{2\pi\rho r}\left(\frac{dB_{0\varphi}}{dr} - \frac{B_{0\varphi}}{r}\right)\Big|_{r=R_0}\right]\frac{k^2d^2}{\pi^2 + k^2d^2}. \quad (3.3)$$

For stability it is necessary and sufficient that $\omega^2 > 0$. In the particular case of a power dependence of the field on the distance to the axis,

$$B_{0\varphi} = B_{0\varphi}(r/R_0)^n \qquad (n \ll R_0/d) \quad (3.4)$$

and relation (3.3) becomes

$$\omega^2 = \left[4a\Omega_0 - \frac{B_{0\varphi}^2(n-1)}{2\pi\rho R_0^2}\right]\frac{k^2d^2}{\pi^2 + k^2d^2}. \quad (3.5)$$

The faster the field drops off to the periphery, the greater the stabilizing effect of the field.

A more graphical way of obtaining the same result is by the Rayleigh method. An element rotating in some layer, is "glued" to the corresponding lines of force. If it leaves the layer, no force arises along φ. Therefore, as in the absence of a field, the axial component of the angular momentum of the element is conserved. However, the lines of force are distorted, and the entire force tube is dragged into motion along with the element. If the pressure gradient in the external layer cannot equalize the centrifugal force $\rho\Omega^2 r$, i.e., if $\Omega d(\Omega r^2)/dr < 0$, the force tube will be accelerated in the direction of motion. But the volume of the stretched tube is conserved, while its radius decreases as $1/r$. Because of the conservation of the magnetic flux, the field in the tube increases as r. If the equilibrium field grows faster, then the decrease in the strong field entering the tube from the periphery is not compensated by the growth of the weak field in the tube coming to replace it from the inside. Part of the magnetic energy changes into disturbance energy.

If the equilibrium field increases as r, then the transfer of the force tubes does not influence the development of the instability.

If the field decreases from the axis or grows slower than r then, for radial circulation, the perturbation energy changes into energy of the magnetic field. In a sufficiently strong field the rate at which the tube gains energy from the centrifugal force is less than the rate of growth of the magnetic energy. The tube will stop, having used up all of the priming supply. On the other hand, if this supply is sufficient for the tube to reach the outer wall, there will be no further growth in average magnetic energy. Therefore the flow is stable only in linear approximation, i.e., one possible stationary flow is separated from the other by a potential barrier.

In this manner, the decrease in the velocity and the growth of the field bring about instability. It is interesting to evaluate their respective influences. For this, following Shafranov,[7] let us examine a flow of the form

$$B_0 = \beta\sqrt{4\pi\rho}V_0. \quad (3.6)$$

From (3.3) it follows that

$$\omega^2 = 2(\Omega_1 - \Omega_2)V_0(\beta^2 - 1)k^2d/(\pi^2 + k^2d^2), \quad (3.7)$$

i.e., the flow is stable if $\Omega_1 > \Omega_2$, when $\beta^2 < 1$, and if $\Omega_1 < \Omega_2$ when $\beta^2 > 1$. $\beta^2 = 1$ is a critical value at which the effects of the field and the velocity are balanced.

In conclusion let us point out that conditions (2.15), (2.16), limiting the region of applicability of the results, are fulfilled both in experiments with certain liquid metals (Na) and under astrophysical conditions.

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CHARGED PARTICLE ENERGY LOSSES DUE TO EXCITATION OF PLASMA OSCILLATIONS

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Beam electrons and plasma oscillations are regarded as two subsystems. A kinetic equation describing the interaction between the beam and plasma is obtained on the assumption that the beam does not change the properties of the plasma and that the plasma state is specified by its equilibrium parameters. The expression for the decelerating force calculated on the basis of this equation includes losses due to electron-electron collisions as well as those due to the excitation of plasma oscillations. A more general case is considered in which neither of the subsystems is in thermal equilibrium. The solution of a set of nonlinear equations for the beam electron distribution function and the electric potential is considered for this particular case. The results are used to account for the rapid energy transfer from beam electrons to plasma electrons, which was first observed by Langmuir.

IN calculating the energy losses of electrons moving through a plasma it is customary to consider separately the losses resulting from short-range interactions (electron-electron collisions) and those resulting from the excitation of plasma oscillations.

The calculation for electron collisions results in the following expression for the decelerating force:

$$F_1 = (e\omega_L/v_0)^2 \ln (r_d/a), \qquad (1)$$

where e is the electron charge, $\omega_L = \sqrt{4\pi e^2 n/m}$ is the Langmuir frequency, v_0 is the velocity of electrons entering the plasma, $a = e^2/mv_0^2$ and r_d is the Debye shielding distance. Equation (1) can be obtained from Landau's kinetic equation[1] or from the corresponding Fokker-Planck equation, where (1) represents the systematic frictional force exerted by plasma electrons on a beam electron.[1,9]

The decelerating force resulting from the distant part of the interaction is usually calculated in the approximation of the given particle motion. The following expression is obtained:[2-8]*

$$F_2 = (e\omega_L/v_0)^2 \ln (v_0/v_T). \qquad (2)$$

It follows from (1) and (2) that the energy losses resulting from near and distant interactions are of the same order of magnitude.

*The results obtained by various workers differ in the logarithmic term.

It was established very early by Langmuir that an electron beam in a plasma is scattered much more rapidly than (1) and (2) indicates. Langmuir suggested that this extremely rapid scattering is associated with the excitation of plasma oscillations; this was later confirmed experimentally (see reference 10, for example).

The problem has been investigated theoretically in papers by Vlasov,[2] Bohm and Gross[3] and others. In these papers it is assumed that the velocities of electrons entering the plasma are modulated as they traverse the double space-charge sheath and that the electrons then form bunches of different densities, as in a klystron oscillator. Regions of maximum beam density are also regions of strong scattering. However, as will be seen below, there is considerable analogy with the operation of a traveling-wave tube.[11] We shall now briefly indicate the results obtained in the two parts of the present work.

Following Bohm and Pines,[12] the Hamiltonian for beam and plasma electrons is

$$H = \frac{1}{2m} \sum_{1 \leqslant i \leqslant N} \left(\mathbf{P}_i - \frac{e}{c} \mathbf{A}(\mathbf{q}_i) \right)^2 + \frac{1}{8\pi} \int E^2 \, d\mathbf{q}, \qquad (3)$$

where $\mathbf{A}$ is the vector potential of the longitudinal electric field ($\mathbf{E} = -(1/c)\,\partial \mathbf{A}/\partial t$; curl $\mathbf{A} = 0$), and N is the number of electrons in the system. With $\mathbf{A}$ as the field coordinate we have the momentum $\Pi = -\mathbf{E}/4\pi c$. The Fourier series for $\mathbf{A}$ and Π are

$$A = \sqrt{\frac{4\pi c^2}{V}} \sum_{k,j} a_k Q_k^{(j)} \begin{matrix}\sin kq \\ \cos kq\end{matrix}, \quad \Pi = \frac{1}{\sqrt{4\pi c^2 V}} \sum_{k,j} a_k P_k^{(j)} \begin{matrix}\sin kq \\ \cos kq\end{matrix}. \quad (4)$$

Here a_k is a unit vector. In (4) and hereinafter the upper function pertains to $j = 1$ and the lower function to $j = 2$.

The state variables of the system will be the electron coordinates and momenta, q_i and P_i, and the coordinates and momenta, $Q_k^{(j)}$ and $P_k^{(j)}$, of plasma oscillators with wave numbers $k < k_d$. Here $k_d \approx 1/r_d$. Substituting (4) into (3) and separating terms with $k < k_d$ from those with $k > k_d$, we obtain the following Hamiltonian in linear approximation:

$$H = \sum_i \frac{P_i^2}{2m} - \frac{e}{m}\sqrt{\frac{4\pi}{V}} \sum_{i,k<k_d} (P_i a_k)(Q_k^{(1)} \sin kq_i + Q_k^{(2)} \cos k q_i) + \frac{1}{2} \sum_{k<k_d,j} (P_k^{(j)2} + \omega_L^2 Q_k^{(j)2}) + \frac{1}{2} \sum_{i,j} U(|q_i - q_j|). \quad (5)$$

Here the first term represents the kinetic energy of the electrons, the third term is the energy of the plasma oscillations with frequency ω_L and $k < k_d$, the second term is the interaction energy of plasma oscillations and electrons, and the last term represents the screened (near) part of the electron interaction energy.

We introduce the distribution function of electrons and plasma oscillations $f(q_i, P_i, Q_k^{(j)}, P_k^{(j)}, t)$, which specifies the probabilities of different states of the system. By means of (5) we obtain the following expression for f:

$$\frac{\partial f}{\partial t} + \sum_i \left\{ \frac{P_i}{m} - \frac{e}{m}\sqrt{\frac{4\pi}{V}} \sum_{k,j} a_k Q_k^{(j)} \begin{matrix}\sin kq_i \\ \cos kq_i\end{matrix} \right\} \frac{\partial f}{\partial q_i} + \sum_{k,j} \left\{ P_k^{(j)} \frac{\partial f}{\partial Q_k^{(j)}} - \omega_L^2 Q_k^{(j)} \frac{\partial f}{\partial P_k^{(j)}} \right\} - \sum_i \frac{\partial}{\partial q_i} \sum_j U(|q_i - q_j|) \frac{\partial f}{\partial P_i} \pm \frac{e}{m}\sqrt{\frac{4\pi}{V}} \sum_{i,k,j} \left\{ (P_i a_k) k Q_k^{(j)} \begin{matrix}\sin kq_i \\ \sin kq_i\end{matrix} \frac{\partial f}{\partial P_i} + (P_i a_k) \begin{matrix}\sin kq_i \\ \cos kq_i\end{matrix} \frac{\partial f}{\partial P_k^{(j)}} \right\} = 0. \quad (6)$$

In the first part of the present paper we obtain from (6) approximate kinetic equations for the electron distribution function $f_1(q, P, t)$ and for $F_1(Q_k^{(i)}, P_k^{(i)}, t)$, which is the coordinate and momentum distribution of plasma oscillations with the wave vector k.

The kinetic equation obtained for f_1 differs from the familiar equation of Landau by taking the excitation of plasma waves into account besides electron-electron collisions. In linear approximation this becomes the Fokker-Planck equation in momentum space, with the systematic frictional term consisting of two parts corresponding to (1) and (2) for the decelerating force.

The systematic frictional term in the kinetic equation for F_1 corresponds to the damping coefficient of plasma oscillations obtained by Landau.[13]

As already noted, under certain conditions the transfer of energy from nonequilibrium electrons to plasma electrons occurs at distances considerably smaller than the relaxation lengths obtained by means of (1) and (2). The existence of this Langmuir effect indicates that the kinetic equation used in the first part of the present paper does not determine electron deceleration in all cases. In deriving this kinetic equation for the electrons we assume equilibrium states of the plasma electrons surrounding a given beam electron and of the plasma oscillations. However, with a sufficiently high concentration of nonequilibrium electrons (such as beam electrons entering the plasma) these conditions are not satisfied and a set of simultaneous nonlinear equations for the beam and plasma must be solved to determine the beam deceleration. This will be done in the second part of the present paper.

1. DERIVATION OF KINETIC EQUATIONS FOR f_1 AND F_1

When (6) is integrated over all variables except the coordinate and momentum of a single particle, and then over all variables except the coordinates and momentum of a single oscillator with wave number k, we obtain the first two of a chain of equations for the distribution functions:

$$\frac{\partial f_1}{\partial t} + \frac{P}{m}\frac{\partial f_1}{\partial q} - \frac{e}{m}\sqrt{\frac{4\pi}{V}} \int \sum_{k,j} a_k Q_k^{(j)} \begin{matrix}\sin kq \\ \cos kq\end{matrix} \frac{\partial \Phi_2}{\partial q} dQ_k^{(j)}\, dP_k^{(j)} \pm \frac{e}{m}\sqrt{\frac{4\pi}{V}} \sum_{k,j} \int (P a_k) k Q_k^{(j)} \begin{matrix}\cos kq \\ \sin kq\end{matrix} \frac{\partial \Phi_2}{\partial P} dQ_k^{(j)}\, dP_k^{(j)} - n \frac{\partial}{\partial q} \int U(|q - q'|) \frac{\partial f_2}{\partial P} dq'\, dP' = 0, \quad (7)$$

$$\frac{\partial F_1}{\partial t} + \sum_j P_k^{(j)} \frac{\partial F_1}{\partial Q_k^{(j)}} - \sum_j \omega_L^2 Q_k^{(j)} \frac{\partial F_1}{\partial P_k^{(j)}} + n \frac{e}{m}\sqrt{\frac{4\pi}{V}} \int \sum_j (P a_k) \begin{matrix}\sin kq \\ \cos kq\end{matrix} \frac{\partial \Phi_2}{\partial P_k^{(j)}} dq\, dP = 0. \quad (8)$$

Here $\Phi_2(q, P, Q_k^{(j)}, P_k^{(j)}, t)$ is the second mixed distribution function, $f_2(q, P, q', P', t)$ is the second electron distribution function and $n = N/V$ is the average number of electrons per unit volume.

Equations (7) and (8) relate the first and second distribution functions. In a similar manner we can obtain equations for the three functions Φ_2, f_2, F_2 (second distribution functions).

The equations for the second distribution functions contain third distribution functions etc. In

order to obtain an approximate closed set of equation we follow Bogolyubov and Gurov[14] in introducing the following approximate distribution functions:

$$f_3 = f_1 f_1 f_1, \quad \Phi_3 = f_1 f_1 F_1 \text{ and so on} \tag{9}$$

$$f_2 = f_1 f_1 + G(\mathbf{q}, \mathbf{P}, \mathbf{q}', \mathbf{P}', t) \tag{10}$$

$$\Phi_2 = f_1 F_1 + g(\mathbf{q}, \mathbf{P}, Q_{\mathbf{k}}^{(j)}, P_{\mathbf{k}}^{(j)}, t), \tag{11}$$

where G and g are correlation functions which are proportional to a small parameter (the ratio of the interaction energy to the kinetic energy).

We shall first consider the case $f_1(\mathbf{q}, \mathbf{P}, t) = f_1(\mathbf{P}, t)$, i.e., a uniform first distribution function of the electrons. In this approximation, using the approximations introduced above for the distribution functions, from (7), (8) and the corresponding equations for the second distribution functions we obtain the following equations for f_1, F_1, g, and G:

$$\frac{\partial f_1}{\partial t} \pm \frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_{\mathbf{k},j}\int (\mathbf{P}\mathbf{a}_{\mathbf{k}})\,\mathbf{k} Q_{\mathbf{k}}^{(j)} \begin{matrix}\cos \mathbf{kq} \\ \sin \mathbf{kq}\end{matrix} F_1 dQ_{\mathbf{k}}^{(j)} dP_{\mathbf{k}}^{(j)} \frac{\partial f_1}{\partial \mathbf{P}} = \frac{e}{m}\sqrt{\frac{4\pi}{V}}$$
$$\times \sum_{\mathbf{k},j}\int \left\{\mathbf{a}_{\mathbf{k}} Q_{\mathbf{k}}^{(j)} \begin{matrix}\sin \mathbf{kq} \\ \cos \mathbf{kq}\end{matrix} \frac{\partial g}{\partial \mathbf{q}} \mp \mathbf{k}(\mathbf{P}\mathbf{a}_{\mathbf{k}}) Q_{\mathbf{k}}^{(j)} \begin{matrix}\cos \mathbf{kq} \\ \sin \mathbf{kq}\end{matrix} \frac{\partial g}{\partial \mathbf{P}}\right\} dQ_{\mathbf{k}}^{(j)} dP_{\mathbf{k}}^{(j)}$$
$$+ n \frac{\partial}{\partial \mathbf{q}}\int U(|\mathbf{q}-\mathbf{q}'|)\frac{\partial G}{\partial \mathbf{P}} d\mathbf{q}'\, d\mathbf{P}'; \tag{12}$$

$$\frac{\partial F_1}{\partial t} + \sum_j \left(P_{\mathbf{k}}^{(j)} \frac{\partial F_1}{\partial Q_{\mathbf{k}}^{(j)}} - \omega_L^2 Q_{\mathbf{k}}^{(j)} \frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}}\right)$$
$$+ n \frac{e}{m}\sqrt{\frac{4\pi}{V}}\int (\mathbf{P}\mathbf{a}_{\mathbf{k}})^{\sin \mathbf{kq}}_{\cos \mathbf{kq}} f_1 d\mathbf{q}\, d\mathbf{P} \frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}}$$
$$= -n \frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_j \int (\mathbf{P}\,\mathbf{a}_{\mathbf{k}})^{\sin \mathbf{kq}}_{\cos \mathbf{kq}} \frac{\partial}{\partial P_{\mathbf{k}}^{(j)}} g\, d\mathbf{q}\, d\mathbf{P}, \tag{13}$$

$$\frac{\partial G}{\partial t} + \frac{\mathbf{P}}{m}\frac{\partial G}{\partial \mathbf{q}} + \frac{\mathbf{P}'}{m}\frac{\partial G}{\partial \mathbf{q}'}$$
$$= \frac{\partial}{\partial \mathbf{q}} U(|\mathbf{q}-\mathbf{q}'|)\left\{\frac{\partial f_1}{\partial \mathbf{P}} f_1 - f_1 \frac{\partial f_1}{\partial \mathbf{P}'}\right\} = 0; \tag{14}$$

$$\frac{\partial g}{\partial t} + \frac{\mathbf{P}}{m}\frac{\partial g}{\partial \mathbf{q}} + \sum_j \left\{P_{\mathbf{k}}^{(j)} \frac{\partial g}{\partial Q_{\mathbf{k}}^{(j)}} - \omega_L^2 Q_{\mathbf{k}}^{(j)} \frac{\partial g}{\partial P_{\mathbf{k}}^{(j)}}\right\}$$
$$= \mp \frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_j (\mathbf{P}\mathbf{a}_{\mathbf{k}})\,\mathbf{k} Q_{\mathbf{k}}^{(j)} \begin{matrix}\cos \mathbf{kq} \\ \sin \mathbf{kq}\end{matrix} \frac{\partial f_1}{\partial \mathbf{P}} F_1$$
$$- \frac{e}{m}\sqrt{\frac{4\pi}{V}}\sum_j (\mathbf{P}\,\mathbf{a}_{\mathbf{k}})^{\sin \mathbf{kq}}_{\cos \mathbf{kq}} \frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}} f_1. \tag{15}$$

If the initial distributions for G and g are known, we obtain equations for f_1 and F_1 by solving (14) and (15) and eliminating G and g from (12) and (13). Usually only the initial values of f_1 and F_1 are known, in which case, as in reference 15, we can obtain approximate kinetic equations for f_1 and F_1, which are valid only in such large time intervals that the initial values of G and g are no longer significant.

The solution of (14) in this approximation can be represented by

$$G = \frac{\partial}{\partial \mathbf{q}}\int_0^\infty U\left(\left|\mathbf{q}-\mathbf{q}' - \frac{(\mathbf{P}-\mathbf{P}')}{m}\tau\right|\right) d\tau \left\{\frac{\partial f_1}{\partial \mathbf{P}} f_1 - f_1 \frac{\partial f_1}{\partial \mathbf{P}'}\right\}. \tag{16}$$

By using (16) to eliminate G from the last term of (12), we obtain an expression corresponding to the right-hand side of Eq. (10.21) in Bogolyubov's book.[15] This is the approximate kinetic equation for a system of particles with Coulomb interaction, which was derived by Landau. The only difference is that in our case the expansion of the interaction potential energy contains only terms with wave numbers $k > k_d$. Therefore by linearizing this term and assuming that f_1 differs very little from a Maxwellian distribution, we obtain two terms describing diffusion and systematic friction. The corresponding coefficients are finite for small k since the potential energy $U(|\mathbf{q}-\mathbf{q}'|)$ contains only terms with $k > k_d$ in the expansion according to wave numbers. For large k the region of integration is limited by the condition $k \sim mv_0^2/e^2$. The resulting decelerating force agrees with (1).

In order to obtain a closed equation for f_1 we must eliminate the correlation function g from (12). To obtain an equation for f_1 that is accurate up to quadratic terms in the ratio of potential energy to kinetic energy, we may substitute for F_1 in (15) the equilibrium distribution for oscillations:

$$F_1^{(0)} = A \exp\{-P_{\mathbf{k}}^{(j)2}/2\varkappa T - \omega_L^2 Q_{\mathbf{k}}^{(j)2}/2\varkappa T\}. \tag{17}$$

The solution for g then becomes

$$g = \mp \frac{e}{m}\sqrt{\frac{4\pi}{V}}\int_0^\infty \sum_j (\mathbf{P}\mathbf{a}_{\mathbf{k}})\,\mathbf{k}\left(Q_{\mathbf{k}}^{(j)} \cos \omega_L \tau - \frac{P_{\mathbf{k}}^{(j)}}{\omega_L}\sin \omega_L \tau\right)$$
$$\times \begin{matrix}\cos \\ \sin\end{matrix}\left[\mathbf{k}\left(\mathbf{q} - \frac{\mathbf{P}}{m}\tau\right)\right] d\tau F_1^{(0)} \frac{\partial f_1}{\partial \mathbf{P}} + \frac{e}{m}\sqrt{\frac{4\pi}{V}}\frac{1}{\varkappa T}\int_0^\infty \sum_j (\mathbf{P}\,\mathbf{a}_{\mathbf{k}})$$
$$\times \begin{matrix}\sin \\ \cos\end{matrix}\left[\mathbf{k}\left(\mathbf{q} - \frac{P}{m}\tau\right)\right]\{P_{\mathbf{k}}^{(j)} \cos \omega_L \tau$$
$$+ Q_{\mathbf{k}}^{(j)} \omega_L \sin \omega_L \tau\}\, d\tau F_1^{(0)} f_1. \tag{18}$$

Substituting (18) into (12) and integrating over $Q_{\mathbf{k}}^{(j)}$ and $P_{\mathbf{k}}^{(j)}$, we obtain the following kinetic equation for f_1:

$$\frac{\partial f_1}{\partial t} = n \frac{\partial}{\partial \mathbf{q}}\int U(|\mathbf{q}-\mathbf{q}'|)\frac{\partial G}{\partial \mathbf{P}} d\mathbf{q}'\, d\mathbf{P}'$$
$$+ \frac{\partial}{\partial P_\alpha} D_{\alpha\beta} \frac{\partial f_1}{\partial P_\beta} + \frac{\partial}{\partial \mathbf{P}}(\mathbf{A} f_1). \tag{19}$$

Here G is given by (16); for the diffusion coefficient $D_{\alpha\beta}$ and for the coefficient A of systematic friction due to the excitation of random plasma oscillations we obtain

$$D_{\alpha\beta} = \frac{e^2 \varkappa T}{2\pi} \int \delta(\omega_L - \mathbf{kP}/m)\, a_{k_\alpha} a_{k_\beta}\, d\mathbf{k}, \qquad (20)$$

$$\mathbf{A} = \frac{e^2}{2\pi m} \int \mathbf{a}_{\mathbf{k}} (\mathbf{P}\mathbf{a}_{\mathbf{k}})\, \delta(\omega_L - \mathbf{kP}/m)\, d\mathbf{k}. \qquad (21)$$

The decelerating force F_2 is obtained from (21)

Transforming to spherical coordinates in (21) with the z axis along P, we obtain

$$F_2 = e^2 \omega_L \int_{-1}^{1} \int_{0}^{k_d} k\, dk\, \delta\left(\omega_L - \frac{kP}{m} y\right) y\, dy, \qquad y = \cos\theta. \qquad (22)$$

It follows from (22) that the decelerating force F_2 differs from zero only when $P/m \geq \omega_L/k$, which indicates that particles with the momentum P can excite waves only with the wave number $k > \omega_L m/P$. We therefore have

$$F_2 = \frac{e^2 \omega_L^2}{v^2} \int_{\omega_L/v}^{k_d} \frac{dk}{k} = \frac{e^2 \omega_L^2}{v^2} \ln \frac{v}{v_T}. \qquad (23)$$

Here $v = P/m$ and $v_T = r_d/\omega_L$ is the thermal velocity. The total force of systematic friction is given by the sum of F_1 and F_2 and does not depend on the choice of k_d. (Compare with the corresponding results obtained by Vlasov in reference 2.)

By using (7) and the corresponding equation for the second distribution functions we can obtain a kinetic equation for f_1 in the inhomogeneous case. This complicated equation will not be presented here, but in the second part of the present paper we shall use a specific example to show that under certain conditions when the inhomogeneity of the distribution function is taken into account the decelerating force acting on beam electrons can be considerably greater than would follow (1) and (23).

We shall now consider the diffusion coefficient. The only nonvanishing terms in (20) are those with $\alpha = \beta$. With the z axis along P, the integral gives

$$D_{33} = \frac{e^2 \varkappa T}{v^3} \omega_L^2 \ln \frac{v}{v_T},$$

$$D_{11} = D_{22} = \frac{m e^2 \omega_L^2}{4v} \quad \text{for } v \gg v_T. \qquad (24)$$

In reference 9 Temko has calculated the diffusion coefficients associated with the screened part of the interaction.

We note that in the stationary case the kinetic equation (19) is satisfied by a Maxwellian distribution.

We shall now consider the kinetic equation for the coordinate and momentum distribution function of plasma oscillations, which is obtained under the aforementioned assumptions by eliminating the correlation function g from (13) and (15). We insert the equilibrium value of f_1 (the Maxwellian distribution) in the right-hand side of (15). The calculation gives the following equation for F_1:

$$\frac{\partial F_1}{\partial t} + \sum_j \left\{ P_{\mathbf{k}}^{(j)} \frac{\partial F_1}{\partial Q_{\mathbf{k}}^{(j)}} - \left[\omega_{\mathbf{k}}^2 Q_{\mathbf{k}}^{(j)} + \varkappa T \frac{\omega_{\mathbf{k}}^2}{\omega_L^2} \frac{\partial}{\partial Q_{\mathbf{k}}^{(j)}} \right] \frac{\partial F_1}{\partial P_{\mathbf{k}}^{(j)}} \right\}$$
$$= 2\gamma \varkappa T \sum_j \frac{\partial^2 F_1}{\partial P_{\mathbf{k}}^{(j)2}} + 2\gamma \sum_j \frac{\partial}{\partial P_{\mathbf{k}}^{(j)}} (P_{\mathbf{k}}^{(j)} F_1). \qquad (25)$$

Here

$$\omega_k^2 = \omega_L^2 + \frac{3\varkappa T}{m} k^2, \qquad \gamma = \sqrt{\frac{\pi}{8}} \frac{\omega_L}{r_d^3 k^3} \exp(-1/2 r_d^2 k^2). \qquad (26)$$

From (25) we obtain an equation for the amplitudes $Q_{\mathbf{k}}^{(j)}$, averaged by means of the distribution function F_1, of plasma oscillations with different numbers:

$$\ddot{\bar{Q}}_{\mathbf{k}}^{(j)} + 2\gamma \dot{\bar{Q}}_{\mathbf{k}}^{j} + \omega_{\mathbf{k}}^2 \bar{Q}_{\mathbf{k}}^{(j)} = 0, \qquad \bar{Q}_{\mathbf{k}} = \int Q_{\mathbf{k}} F_1 dQ_{\mathbf{k}}\, dP_{\mathbf{k}}. \qquad (27)$$

The damping coefficient γ of plasma oscillations agrees with that calculated by Landau.[13] We also obtain various statistical parameters of plasma oscillations from (25).

2. NONLINEAR THEORY OF PLASMA OSCILLATIONS EXCITED BY AN ELECTRON BEAM

The kinetic equations for f_1 and F_1 were derived on the assumption that at initial time the plasma oscillations (in the case of f_1) or the electrons (in the case of F_1) are in thermal equilibrium. There are many problems in which this is not the case and neither of the subsystems (electrons and plasma oscillations) is in thermal equilibrium. We shall now consider one of these problems.

An electron beam enters the plasma in the x direction at the point $x = 0$ with a velocity that exceeds the thermal velocity of plasma electrons. We shall show that the decelerating force acting on the beam electrons due to the excitation of plasma oscillations is considerably greater than that given by (1) and (2).

We at the very start separate beam electrons and plasma electrons in the Hamiltonian (5). Then the two equations (7) and (8) for the first distribution functions are supplemented by another equation for the beam electron distribution function.

When determining the decelerating forces in the homogeneous case considered above it was important to take into account the correlation between plasma electron variables and the variables pertaining to plasma oscillations, since in this homogeneous case the self-consistent term vanishes.

In the inhomogeneous case, with organized oscillations of the entire system excited at the expense of electron beam energy, in approximating

the second distribution function of electrons and plasma oscillations we may set

$$\Phi_2(\mathbf{q}, \mathbf{P}, Q_k^{(j)}, P_k^{(j)}, t) = f_1 F_1, \quad (28)$$

since in the inhomogeneous case this multiplicative term is the principal term.

Taking (28) for the second distribution function, we arrive at a system of self-consistent equations for the distribution functions of electrons and plasma oscillations. We do not present these equations because, farther along, in order to simplify a comparison of our results with those of other writers, we shall use a system of self-consistent equations for the electron distribution function and scalar electric potential which were first investigated by Vlasov and thereafter by many other writers:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial v} = 0, \quad (29)$$

$$\frac{\partial^2 \varphi(x,t)}{\partial x^2} = 4\pi e\left\{\int_{-\infty}^{\infty} f(x,v,t)\,dv - n_+\right\}. \quad (30)$$

We assume that the charge of the electrons is neutralized by the positive ion background.

In solving our problem of electron deceleration through the excitation of plasma waves we must obtain a wave solution of (29) and (30) which satisfies the given boundary conditions at $x = 0$. If φ is assumed to be a known function, by solving (29) with respect to φ and eliminating f from (30) we obtain an equation for the electric potential.

From the solution of the linearized equations (29) and (30) it follows[3,11,16-18] that longitudinal plasma waves are generated growing in the x direction, the phase velocity of which is smaller than the average velocity v of the beam electrons. The rate of growth of the plasma waves depends on the velocity and concentration of the beam electrons, and with a sufficiently small concentration the growth may be as small as desired.

For a sufficiently slow growth of plasma waves the solution for the potential in nonlinear approximation can be obtained in the form

$$\varphi(x,t) = \varphi_0(x)\sin(\omega t - kx + \Psi(x)), \quad (31)$$

where $\varphi_0(x)$ and $\Psi(x)$ are the slowly varying amplitude and phase, respectively.

For a steady wave, i.e., the amplitude and phase are independent of x, the potential is a function of only $x - v_{ph}t$. Then the solution of (29) with respect to φ becomes

$$f(x,v,t) = \Phi\left(\pm\left[(v-v_{ph})^2 - 2\frac{e}{m}\varphi(x-v_{ph}t)\right]^{1/2} + v_{ph}\right), \quad (32)$$

where Φ is an arbitrary function; the + sign is taken for $v > v_{ph}$ and the − sign for $v < v_{ph}$. Eliminating f from (30) by means of (32), we obtain the following equation for φ:

$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi e\left\{\int \Phi(v)\left[1 + \frac{2e\varphi(x - v_{ph}t)}{m(v-v_{ph})^2}\right]^{-1/2} dv - n_+\right\}. \quad (33)$$

With a Maxwellian distribution used for Φ, (33) agrees with the equation given by Bohm and Gross.[3]

Akhiezer and Lyubarskiĭ[19] have solved an equation similar to (33)* for zero temperature of plasma and beam electrons. In another paper Akhiezer, Lyubarskiĭ, and Faĭnberg[20] have solved the more general equation for nonzero plasma temperature. In this case Φ may be represented by

$$\Phi = \Phi_0(mv^2/2) + n_1\delta(v - \bar{v}). \quad (34)$$

Here n_1 is the beam electron concentration, $\bar{v}$ is the electron velocity and Φ_0 is an arbitrary function of the energy.

These solutions cannot be used directly in the problem of electron beam deceleration, since with small thermal losses the transfer of energy from the beam to the wave occurs in the region of wave buildup. We must therefore consider the process whereby the wave is established.

It also remains an open question whether the solution for a growing wave will approach a solution satisfying (33). Yet from this particular solution we can infer that different conditions govern the application of the linear approximation to plasma and beam electrons. Indeed, when Φ in the right-hand side of (33) is replaced by (34) or a more general expression allowing for the thermal spread of beam electrons, it is easily seen that when $v_{ph} \gg \sqrt{\kappa T/m}$ the linear approximation is valid for plasma electrons if $e\varphi \ll mv_{ph}^2/2$ and for beam electrons if $e\varphi \ll m(v_{ph} - \bar{v})^2/2$. When $m(v_{ph} - \bar{v})^2 \ll mv_{ph}^2$ nonlinear effects will be manifested for beam electrons at considerably lower potentials than for plasma electrons. The use of the nonlinear equation only for plasma electrons can, of course, be justified only if the solution yields a steady value of the amplitude such that $e\varphi_{st} \ll mv_{ph}^2/2$.

Thus for a low concentration of beam electrons (but sufficiently large to maintain plasma oscillations at the expense of beam energy), we can replace (29) and (30) by a set of equations for the beam electron distribution f_1 alone, with φ given by the wave equation for the plasma wave. The phase velocity and damping coefficient of the plasma wave will then be taken from the linear theory of plasma oscillations.[2,13]

*The equations differ because of different constants of integration.

We thus arrive at the following set of equations:

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{e}{m}\frac{\partial \varphi}{\partial x}\frac{\partial f_1}{\partial v} = 0, \tag{35}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + 2\frac{\gamma}{v_\phi}\frac{\partial \varphi}{\partial x} - \frac{1}{v_\phi^2}\frac{\partial^2 \varphi}{\partial t^2} = 4\pi e\left\{\int f_1 dv - n_{+1}\right\}, \tag{36}$$

$$\varphi = \varphi_0(x)\sin(\omega_k t - kx + \Psi(x)). \tag{37}$$

Here ω_k has its previous meaning given in (26); γ is the damping coefficient, which includes both that obtained by Landau [see (26)] and possible damping resulting from collisions. The functions $\varphi_0(x)$ and $\Psi(x)$ are, as previously, the slowly varying amplitude and phase.

With (37) for the potential, we solve (35) for a given beam electron distribution function at $x = 0$. We denote the known function f_1 at $x = 0$ by $f_1^{(0)}(v^{(0)})$, with $\int f_1^{(0)} dv^{(0)} = n_{+1}$.* The superscript 0 pertains to the point $x = 0$.

Since, for a given φ, (35), is a first-order linear differential equation, its solution is determined by the solution of the characteristic equation. In view of the slow amplitude and phase variations this equation can be written as

$$\frac{d^2x}{dt} \approx -\frac{ek}{m}\varphi_0(x)\cos(\omega_k t - kx + \Psi(x)). \tag{38}$$

When the functions $v^{(0)}(t, x, v)$ and $t^{(0)}(t, x, v)$ are obtained from (38), the solution of (35) can be written as

$$f_1(x, v, t) = f_1^{(0)}(v^{(0)}(x, v, t)).$$

Using this solution, we obtain the following expressions for the density and current:

$$\rho = -e\int f_1^{(0)}(v^{(0)}(x, v, t))\, dv;$$

$$j = -e\int v f_1^{(0)}(v^{(0)}(x, v, t))\, dv.$$

Substituting this expression for ρ into the right-hand side of (36), we obtain a nonlinear equation for the potential. For low electron beam intensity the right-hand side of this equation is small; we can therefore apply a familiar method in the theory of nonlinear oscillations to obtain simpler equations for the wave amplitude and phase. This necessitates finding the Fourier components of ρ, assuming the beam amplitude and phase to be constant during integration.

Let

$$\rho = \rho^{(1)}\cos(\omega_k t - kx) + \rho^{(2)}\sin(\omega_k t - kx);$$

then

$$\rho^{(i)} = -\frac{e}{\pi}\int_0^{2\pi}\int \begin{matrix}\cos\\ \sin\end{matrix} [\omega_k t - kx] f_1^{(0)}(v^{(0)}(x, v, t))\, dv\, d(kx),$$

$$i = 1, 2. \tag{39}$$

Equation (39) can be simplified by using the Liouville theorem $dx\,dv = dx^{(0)}dv^{(0)}$, or since $dx = v\,dt$, $dx^{(0)} = v^{(0)}dt^{(0)}$, we have $dx\,dv = v^{(0)}dt^{(0)}dv^{(0)}$. After the substitution of variables $t, v \to t^{(0)}, v^{(0)}$ in (39) we obtain the following expression for the Fourier components of the density:

$$\rho^{(i)} = -\frac{e}{\pi v_{ph}}\int_0^{2\pi}\int \begin{matrix}\cos\\ \sin\end{matrix} [\omega_k t(t^{(0)}, v^{(0)}, x) - kx] f_1^{(0)}(v^{(0)})\, v^{(0)} dv^{(0)} d(\omega_k t^{(0)}). \tag{40}$$

We can obtain $\rho^{(i)}$ from (40) if $t(t^{(0)}, v^{(0)}, x)$ is known. When we return to the equation of motion (38) and assume that the variable t in the right-hand side of this equation is a known function of the coordinates $v^{(0)}$ and $t^{(0)}$, the energy integral of (38) can be represented as

$$v = v^{(0)}\left\{1 - \frac{2ek}{mv^{(0)2}}\int_0^x \varphi_0(x')\cos[\omega_k t^{(0)} + \omega_k(t - t^{(0)}) - kx' + \Psi(x')]\, dx'\right\}^{1/2},$$

and for the function $t(t^{(0)}, v^{(0)}, x)$ we obtain the integral equation

$$t - t^{(0)} = \frac{1}{v^{(0)}}\int_0^x \left\{1 - \frac{2ek}{mv^{(0)2}}\int_0^{x'} \varphi_0(x'')\cos[\omega_k t^{(0)} + \omega_k(t - t^{(0)}) - kx'' + \Psi(x'')]\, dx''\right\}^{-1/2} dx'. \tag{41}$$

Because of the complexity of (41) we can solve our problem without the use of numerical methods only in certain special cases. We shall now consider some of these cases.

Let μ be a small parameter. We consider the case where the steady amplitude of the oscillations is such that $e\varphi_0/mv^{(0)2} \sim \mu^2$ and the maximum of the excitation pertains to the waves for which $(v^{(0)} - v_{ph})\, v^{(0)} \sim \mu$. Under these conditions and with slowly varying amplitude and phase $\varphi_0(x)$ and $\Psi(x)$, (41) can be simplified by expanding the square root in a series of which only the first two terms are retained. (41) then becomes

$$t - t^{(0)} = \frac{x}{v^{(0)}} + \frac{ek}{mv^{(0)3}}\int_0^x (x - x')\varphi_0(x')\cos[\omega_k t^{(0)} + \omega_k(t - t^{(0)}) - kx' + \Psi(x')]\, dx'. \tag{42}$$

Equation (42) contains two parameters of length. One of these parameters, $A = k(v^{(0)} - v_{ph})/v^{(0)}$ is determined from the excess of the electron stream velocity over the phase velocity of the rapidly growing plasma wave at $x = 0$; the second

*If the electron beam is modulated with respect to density or velocity the boundary form of the distribution will, of course, differ.

parameter α characterizes the rate of change of wave amplitude and phase. Let us now consider the case when $\alpha/A \sim \mu$.

We denote the ratio $e\varphi_0/m\,(v^{(0)}-v_{ph})^2$ by X and obtain an approximate solution of (42) as a power series in X, assuming $X \lesssim 1$. Retaining terms up to X^3 inclusively, we obtain the following expression for $t-t^{(0)}$:

$$\omega(t-t^{(0)}) = \frac{\omega x}{v^{(0)}} - (X - \tfrac{5}{16}X^3)\cos[\omega t^{(0)} - Ax + \Psi] - \tfrac{1}{8}X^2 \sin 2[\omega t^{(0)} - Ax + \Psi]. \quad (43)$$

Substituting this into the integrand of (40), we integrate over $t^{(0)}$, separate the terms in X to X^3 and use the formula

$$\int \frac{f_1(v-\bar{v})}{(v-v_{ph})^n}dv = \oint \frac{f_1(v-\bar{v})}{(v-v_{ph})^n}dv + i\pi\,\frac{1}{(n-1)!}\left(\frac{\partial^n f_1}{\partial v^n}\right)_{v=v_{ph}},$$

in which $\oint$ denotes that the principal value of the integral is taken. In view of (39) we now obtain the following expression for the beam density:

$$\rho = -\frac{e^2}{m}\oint\left[1 - \frac{3}{8}\left(\frac{e\varphi_0}{m(v-v_{ph})^2}\right)^2\left(\frac{v_\Phi}{v}\right)^2\right]\frac{f_1(v-\bar{v})}{(v-v_{ph})^2}dv\cdot\varphi + \frac{\pi e^2}{mk}\left[\left(\frac{\partial f_1}{\partial v}\right)_{v=v_{ph}} - \frac{3}{8\cdot 4!}\left(\frac{e\varphi_0}{m}\right)^2\left(\frac{\partial^5 f_1}{\partial v^5}\right)_{v=v_{ph}}\right]\frac{\partial\varphi}{\partial x}. \quad (44)$$

Substituting this into the right-hand side of (36), we obtain a nonlinear equation for the potential when the solution of this equation is sought in the form (37).

When the ratio of beam and plasma electron concentrations is such that the parameter characterizing slowness of wave amplitude and phase variations is of the same order of magnitude as the parameter characterizing smallness of the right-hand side of the equation for φ, we equate terms of the same order of smallness and obtain the following equations for the wave amplitude and phase:

$$d\varphi_0/dx = \alpha\varphi_0 - \beta\varphi_0^3, \quad (45)$$

$$\frac{d\Psi}{dx} = -\frac{2\pi e^2}{mk}\oint\left[1 - \frac{3}{8}\left(\frac{e\varphi_0}{m(v-v_{ph})^2}\right)^2\left(\frac{v_{ph}}{v}\right)^2\right]\frac{f_1(v-\bar{v})}{(v-v_{ph})^2}dv. \quad (46)$$

The following notation has been used in (45):

$$\alpha = \frac{2\pi^2 e^2}{mk}\left(\frac{\partial f_1}{\partial v}\right)_{v=v_{ph}} - \frac{\gamma}{v_{ph}}, \quad \beta = \frac{3}{8\cdot 4!}\frac{2\pi^2 e^2}{mk}\left(\frac{e}{m}\right)^2\left(\frac{\partial^5 f_1}{\partial v^5}\right)_{v=v_{ph}}.$$

For self excitation of oscillations the coefficient α must be positive. When the damping coefficient γ is given essentially by the damping coefficient of plasma oscillations, i.e., when collisions play a small part, the condition for self excitation becomes

$$(\partial f^{(0)}/\partial v)_{v=v_{ph}} > 0,$$

where $f^{(0)}$ is the distribution function of all electrons (of both the plasma and beam) at $x = 0$. This self-excitation condition corresponds to that given by other authors.[3,16-18]

To a sufficient degree of accuracy, the distribution $f_1^{(0)}$ can now be specified as

$$f_1^{(0)} = n_1(m/2\pi\varkappa T_1)^{1/2}\exp[-m(v-\bar{v})^2/2\varkappa T_1].$$

Here n_1, T_1 and $\bar{v}$ are the concentration, temperature, and velocity of beam electrons. With this boundary distribution function the autoexcitation condition is satisfied for the phase velocity region $\Delta_{ph}v \sim \sqrt{\varkappa T_1/m}$. The coefficient α is maximal for a wave with the phase velocity $\bar{v} - v_{ph} = \sqrt{T_1\varkappa/m}$. Since $\bar{v}$, $v_{ph} \gg \sqrt{\varkappa T_1/m}$ and $v_{ph} = \omega_k/k \approx \omega_L/k$, we obtain $k \approx \omega_L/\bar{v}$ as the wave number of the most rapidly growing wave. For this wave we have

$$\alpha = k\left[\sqrt{\frac{\pi}{22}}\frac{n_1}{n}\frac{\bar{v}^2 m}{\varkappa T_1} - \frac{\gamma}{\omega_L}\right], \quad \beta = \frac{3}{32}\sqrt{\frac{\pi}{22}}\frac{n_1}{n}\frac{\bar{v}^2 m e^2}{(\varkappa T_1)^3}.$$

It follows from the expression for α that for given values of the parameters $\bar{v}$, n, T_1 and γ a lower limit always exists for the concentration of beam electrons which can accompany autoexcitation of oscillations. At lower concentrations an equilibrium velocity distribution for electron motion through the plasma is established only as a result of the relaxation processes described in the first part of the present paper.

The solution of (45) for the amplitude is given by

$$\varphi_0(x) = \varphi^{(0)}e^{\alpha x}\left[1 + \frac{\alpha}{\beta}\,{}^{(0)2}(e^{2\alpha x} - 1)\right]^{-1/2}. \quad (47)$$

Here $\varphi^{(0)}$ is the amplitude at $x = 0$. Let φ_{st} be the steady-state value of the amplitude. For small x ($x \ll 1/\alpha$) the solution of (47) increases exponentially with x. For large x the amplitude approaches

$$\varphi_{st} = \sqrt{\alpha/\beta}. \quad (48)$$

When γ is so small that the second term in the expression for α can be neglected we have $e\varphi_{st} \approx 3\varkappa T_1$.

It is evident from the above equations that the steady-state wave amplitude approaches zero for $T_1 \to 0$ i.e., for a single-velocity beam. It follows from the expression for α that in this case the self-excitation condition for plasma waves is not fulfilled. This does not mean, of course, that plasma waves are not excited when a single-velocity beam passes through a plasma. The solution

being considered here is obtained when $\alpha \ll A$ and $X \lesssim 1$, under which conditions plasma waves actually do not arise in the hydrodynamic approximation. The solution for $\alpha \sim A$ must be considered to describe wave excitation in this case.

We shall now consider (46) for the variation of phase; this equation determines the variation of plasma wave number with increasing x ($\Delta k = -d\Psi/dx$). It follows from (46) that Δk is given by two terms, one of which depends on the amplitude. Both terms are of the order of $\mu^2 k$ and are thus small compared with Δk, defined as the initial difference between the velocity of beam electrons and the wave velocity, which is of the order of μ.

We now estimate the distance in which the energy of beam electrons is transformed into the energy of plasma oscillations. For this purpose we require the ratio of the flux of plasma wave electrical energy in the region where a steady wave has already been established, to the electronic energy flux at $x = 0$. This ratio is represented by

$$S_{wave}/S_{el} = (n/n_1)(e\varphi_0/m\bar{v}^2)^2. \qquad (49)$$

The order of magnitude of the ratio is estimated as follows. Since $e\varphi_0/m\bar{v}^2 \sim \mu^2$ and $\alpha/k \sim \mu^2$, it follows from the expression for α that the ratio of beam and plasma electron concentrations is $n_1/n \sim \mu^4$. From (49) we find that for these values of the parameters the ratio of the energy fluxes is of the order of unity. Thus the beam energy is transformed into plasma wave excitation in the distance l, which is equal in magnitude to the distance within which a wave is established. Denoting the plasma wavelength by λ, we have

$$\lambda \ll l \lesssim 1/\alpha.$$

With currents of 20 — 25 ma, $\kappa T \approx 1$ ev, $m\bar{v}^2/2 \approx 20$ ev, $n_1 = 3 \times 10^8$ sec^{-1} and $n \approx 3 \times 10^{10}$ sec^{-1}, l is of the order of a centimeter. When these same numerical data are used, the relaxation length calculated from (1) and (2) is about 10^5 cm.

Looney and Brown[21] have observed standing waves in a plasma traversed by an electron beam. Standing waves arise when a reflecting electrode is present. In order to determine the conditions for the generation of standing plasma waves by an electron beam the solution for the electric field can be obtained in the form

$$\varphi_s = \varphi_0(t)\sin(\omega t + \Psi(t))\sin(s\pi x/L), \qquad s = 1, 2, \ldots,$$

where $\varphi(t)$ and $\Psi(t)$ are the wave amplitude and phase, which vary slowly with time, and L is the length of the plasma in the direction of beam motion. The calculation shows that for $n_1/n \ll 1$ the conditions for autoexcitation are best satisfied for frequencies and wavelengths given by

$$\omega_L \approx s\pi\bar{v}/L, \qquad \lambda_s \approx 2L/s.$$

It follows that a transition from the fundamental oscillatory mode to higher modes occurs only with increased plasma electron concentration or reduced average velocity of beam electrons.

In the experiments of Looney and Brown the beam electron concentration was the basic factor, i.e., $n_1/n \gg 1$. An analysis of this case will regard the beam as the initial wave system. Under these conditions the square of the oscillatory frequency for a given mean velocity will be proportional to the beam electron concentration or to the current.

Spatial periodicity was detected differently in the well-known work of Merrill and Webb,[10] which we shall not discuss here. We note only that growing plasma waves can be detected by measuring the root-mean-square potential difference between two probes. When one of the probes is moved along the beam axis this quantity will be a periodic function of the probe separation with increasing amplitude. The spatial period of this function is the length of the most rapidly growing plasma wave,

$$\lambda = (2\pi\bar{v}/\omega_L)(1 - \sqrt{\varkappa T_1/m}/\bar{v}).$$

The experiments of Merrill and Webb also indicated the existence of growing plasma waves; the observed spatial periodicity was in good agreement with the value of λ derived from this formula.

We know that plasma waves can arise at the expense of the energy of relative electron-ion motion. The passage of strong current pulses can in this way cause an appreciable rise of the plasma temperature.[22] Plasma waves can be established by pulses of $\sim 10^{-3}$ sec duration. The amount of energy transformed directly into heat will be determined by the damping rate of such waves.

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SCATTERING OF FAST PIONS BY DEUTERONS

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Shadow effects appearing in the diffraction scattering of fast π mesons were examined.

THE experimental data on the scattering of π mesons by protons show that collisions in the high energy region (> 1 Bev) have a diffraction character, that is, the scattering takes place primarily into small angles.[1-4] Belen'kiĭ carried out an analysis of the diffraction scattering of high energy π mesons by protons on the basis of a general theory, not connected with any concrete model of the nucleus.[5,6] Grishin and Saitov made an analogous examination of the diffraction scattering of high energy protons by protons.[7] Lately, Blokhintsev, Barashenkov, and Grishin used the results of an analysis of diffraction scattering of π's on protons to determine the mean radius of the proton and get knowledge of the nucleon structure.[8]

Data on the interactions of π's at high energies with neutrons can be got by studying π scattering by deuterons. For analyzing the diffraction scattering of π's by deuterons one must necessarily consider the shadow effects, whose existence was first demonstrated by Glauber, using the black sphere model of the nucleon.[9] In the present note the diffraction scattering of fast π's by deuterons is examined within the framework of the hypotheses of reference 5.

For high energy incident π's one can assume that the scattering takes place independently from the neutron and proton. In this case the amplitude for elastic π-d scattering can be written in the form

$$f(\vartheta) = \frac{ik}{2\pi}\int \exp(-i\varkappa\rho_d) \times \{1 - S_n(\rho_n)\, S_p(\rho_p)\}\, \varphi^2_0(r)\, dr d\rho_d, \qquad (1)$$

where S_n and S_p are the scattering functions of the on the neutron and proton, $\rho_d = \rho_n + \rho_p/2$ is the plane radius vector from the center of mass of the deuteron, $\mathbf{r} = \mathbf{r}_n - \mathbf{r}_p$ is the relative radius vector, $\kappa = \kappa\theta$ (θ = scattering angle), and $\varphi_0(r)$ is the wave function of the ground state of the deuteron.

The scattering cross section σ_s, the absorption cross section σ_a, and the total cross section σ_t are determined by the following formulas:

$$\sigma_s = \iint |1 - S_n(\rho_n)\, S_p(\rho_p)|^2 \varphi^2_0(r)\, dr d\rho_d,$$
$$\sigma_a = \iint \{1 - |S_n(\rho_n)\, S_p(\rho_p)|^2\}\, \varphi^2_0(r)\, dr d\rho_d,$$
$$\sigma_t = 2\iint \{1 - \mathrm{Re}\, S_n(\rho_n)\, S_p(\rho_p)\}\, \varphi^2_0(s)\, dr d\rho_d. \qquad (2)$$

The cross section σ_s describes the elastic scattering of a π by a deuteron as well as the scattering of a π which accompanies a deuteron aplitting. Analogously the cross section σ_a describes the absorption of a π by a deuteron as well as the absorption of a π accompanying the splitting of a deuteron.

We choose the π-n or π-p scattering functions to be of the form

$$S_n = 1 - \alpha_n \exp(-\rho_n^2/R_n^2),$$
$$S_p = 1 - \alpha_p \exp(-\rho_p^2/R_p^2), \qquad (3)$$

where the parameters α and R must be determined from experiment.

Choosing the wave function for the deuteron ground state in the form of a Gaussian function[10]

$$\varphi_0(r) = Ne^{-\gamma^2 r^2}, \quad N^2 = 8/\pi^3 R_d^3, \quad \gamma = \sqrt{2/\pi}\, R_d^{-1}, \qquad (4)$$

we get for the amplitude of the elastic π-d scattering the expression

$$f(\vartheta) = \tfrac{i}{2}\alpha k R^2 \Big\{\exp[-\tfrac{1}{4}(1+r^2)\varkappa^2 R^2] + \tau\zeta^2 \exp[-\tfrac{1}{4}(\zeta^2+r^2)\varkappa^2 R^2] - \frac{\alpha\tau\zeta^2}{1+\zeta^2+4r^2}\exp\Big[-\frac{1}{4}\frac{\zeta^2+(1+\zeta^2)r^2}{1+\zeta^2+4r^2}\varkappa^2 R^2\Big]\Big\}, \qquad (5)$$

where

$$R = R_p, \quad r^2 = (\pi/16)(R_d/R)^2.$$
$$\tau = \alpha_n/\alpha_p, \quad \zeta = R_n/R_p, \quad \alpha = \alpha_p,$$

The total cross section for the π-d interaction is

$$\sigma_t = 2\pi\alpha R^2 \{1 + \tau\zeta^2 - \alpha\tau\zeta^2/(1 + \zeta^2 + 4r^2)\}. \quad (6)$$

If the parameter r goes to infinity, then

$$\sigma_t = \sigma_t^{(p)} + \sigma_t^{(n)}, \qquad r \gg 1.$$

For finite r the cross section for the π-d interaction is less than the sum of the interaction cross sections for an individual neutron and proton (shadow effect). This effect becomes strongest for small values of the parameter r:

$$\sigma_t = 2\pi\alpha R^2 \left\{1 + \tau\zeta^2 - \frac{\alpha\tau\zeta^2}{1+\zeta^2}\right\}, \qquad r \ll 1.$$

The scattering and absorption π-d cross sections are:

$$\sigma_s = \frac{\pi}{2}\alpha^2 R^2 \left\{1 + \tau^2\zeta^2 + \frac{4\tau\zeta^2}{1+\zeta^2+4r^2} - \frac{4\alpha\tau\zeta^2}{1+2\zeta^2+8r^2} - \frac{4\alpha\tau^2\zeta^2}{2+\zeta^2+8r^2} + \frac{\alpha^2\tau^2\zeta^2}{1+\zeta^2+8r^2}\right\},$$

$$\sigma_a = 2\pi\alpha R^2 \left\{(1 + \tau^2\zeta^2)\left(1 - \frac{\alpha}{4}\right) - \frac{8\alpha\tau\zeta^2}{1+\zeta^2+4r^2} + \frac{\alpha^2\tau\zeta^2}{1+2\zeta^2+8r^2} + \frac{\alpha^2\tau^2\zeta^2}{2+\zeta^2+8r^2} - \frac{\alpha^3\tau^2\zeta^2}{4(1+\zeta^2+8r^2)}\right\}. \quad (7)$$

The angular distribution for π-d scattering (taking into account possible deutron splitting) is determined by the expression

$$\frac{d\sigma_s}{do} = \frac{k^2}{4\pi^2}\int \varphi_0^2(r) \left|\int \exp(-i\varkappa\rho_d)\{1 - S_n(\rho_n) S_p(\rho_p)\}\, d\rho_d\right|^2 dr. \quad (8)$$

Using (3) and (4), we get the following formula for the angular distribution:

$$\frac{d\sigma_s}{do} = \frac{1}{4}\alpha^2 k^2 R^4 \left\{\exp\left[-\frac{1}{2}\varkappa^2R^2\right] + \tau^2\zeta^4 \exp\left[-\frac{1}{2}\zeta^2\varkappa^2R^2\right] + 2\tau\zeta^2 \exp\left[-\frac{1}{4}(1+\zeta^2+4r^2)\varkappa^2R^2\right] - \frac{2\alpha\tau\zeta^2}{1+\zeta^2+4r^2}\exp\left[-\frac{1}{4}\frac{1+2\zeta^2+8r^2}{1+\zeta^2+4r^2}\varkappa^2R^2\right] - \frac{2\alpha\tau^2\zeta^4}{1+\zeta^2+4r^2}\exp\left[-\frac{1}{4}\frac{2+\zeta^2+8r^2}{1+\zeta^2+4r^2}\zeta^2\varkappa^2R^2\right] + \frac{\alpha^2\tau^2\zeta^4}{(1+\zeta^2)(1+\zeta^2+8r^2)}\exp\left[-\frac{1}{2}\frac{\zeta^2}{1+\zeta^2}\varkappa^2R^2\right]\right\}. \quad (9)$$

The parameters α and R can be determined by analyzing the experimental data for π scattering by protons. Then, using the experimental data for π scattering by deuterons (σ_s and σ_t), one can determine the quantities τ and ζ, which characterize the difference between the neutron and corresponding proton parameters. The accuracy of the experimental data at present is, however, not sufficient for such an analysis.

Within the limits of present experimental accuracy one can put $\tau = \zeta = 1$. In that case formulas (6), (7), and (9) are simplified in an essential way:

$$\sigma_t = 2\pi\alpha R^2\left\{2 - \frac{\alpha}{2(1+2r^2)}\right\},$$

$$\sigma_s = \frac{\pi}{2}\alpha^2R^2\left\{2 + \frac{2}{1+2r^2} - \frac{8\alpha}{3+8r^2} + \frac{\alpha^2}{2(1+4r^2)}\right\},$$

$$\sigma_a = 2\pi\alpha R^2\left\{2\left(1 - \frac{\alpha}{4}\right) - \frac{\alpha}{1+2r^2} + \frac{2\alpha^2}{3+8r^2} - \frac{\alpha^3}{8(1+4r^2)}\right\},$$

$$\frac{d\sigma_s}{do} = \frac{1}{4}\alpha^2k^2R^4\left\{2\exp\left[-\frac{1}{2}\varkappa^2R^2\right] + 2\exp\left[-\frac{1}{2}(1+2r^2)\varkappa^2R^2\right] - \frac{2\alpha}{1+2r^2} \times \exp\left[-\frac{1}{8}\frac{3+8r^2}{1+2r^2}\varkappa^2R^2\right] + \frac{\alpha^2}{4(1+4r^2)}\exp\left[-\frac{1}{4}\varkappa^2R^2\right]\right\}. \quad (10)$$

Using the experimental values for the full cross section and the diffraction scattering cross section for 1.4 Bev π's on a proton, $\sigma_t^{(p)} = 34.6 \pm 2.7$ millibarns and $\sigma_s^{(p)} = 7.5 \pm 1.0$ millibarns,[1] we get the parameter values $\alpha = 0.86$, $R = 0.8 \times 10^{-13}$ cm. Using these values and letting $R_d = 2.18 \times 10^{-13}$ cm, we get, for the difference $\Delta_t = \sigma_t - 2\sigma_t^{(p)}$, $\Delta_t = -3.8$ millibarns. The experimental value for this difference for $E_\pi = 1.5$ Bev is $\Delta = -3$ millibarns.[2] For the scattering and absorption cross sections we get, respectively, $\Delta_s = 0.7$ and $\Delta_a = -4.5$ millibarns.

We see that the integral π-d scattering cross section is equal, practically, to double the π-p elastic scattering cross section. The shadow effect actually appears in the π-d absorption cross section.

Figure 1 shows the angular distribution for π-p scattering. The points on the graph correspond to the experimental data for π^- scattering on protons at 1.44 Bev.[3] The cross section is given in millibarns/steradian.

Figure 2 shows the angular distribution for π-d

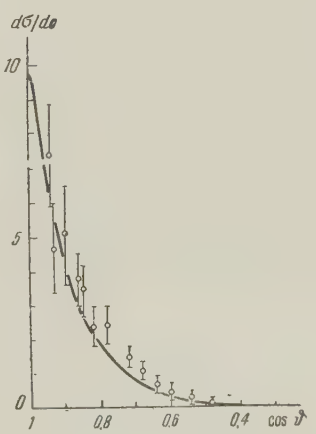

FIG. 1

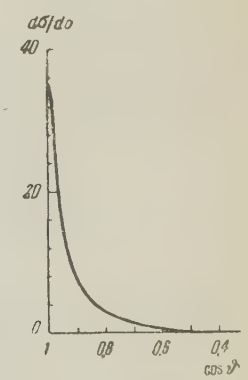

FIG. 2

scattering, calculated according to formula (10).

Obviously, the results obtained can also be applied to the analysis of the high energy scattering of nucleons on deuterons.

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ON THE EVALUATION OF COORDINATE PROBABILITIES FOR NONLINEAR SYSTEMS BY GIBBS' METHOD

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Using the general principles of Gibbs' statistical mechanics, we have developed a method which enables us to evaluate the transition probability density for any generalized coordinate in a system with a nonlinear relaxation mechanism. This method does not require a knowledge of the law of motion for the average value of the coordinate, but uses only the general form of the corresponding equation of motion.

THE method developed by Terletskiĭ and the author[5] using general results obtained earlier[1-4] enables us to evaulate the transition probability density for a generalized coordinate if the behavior of its average value is known when there are additional constant forces present (or included).

The averaged equation of motion contains, however, the average value of the coordinate (the first moment) only when the corresponding system is linear. In the case of a nonlinear system, however, the averaged equation contains also higher moments of the coordinates, the order of which is determined by the character of the nonlinearity.

This fact makes it difficult to use the known nonlinear equations of motion when one wants to find the transition probability density using the scheme given in reference 5, since in problems of Brownian motion one usually starts from the averaged equations of motion and the average dissipative forces produced by the interaction of the system with the medium.

The aim of the present paper consists of setting up a scheme which enables us to evaluate the transition probability density for nonlinear systems starting solely from the general form of the appropriate equation of motion.

1. THE CHARACTERISTIC FUNCTION

As in reference 5, we shall introduce for the generalized coordinate Q the transition probability density $W(Q, t; Q_0, t_0)$ and the characteristic function $Z(a, t; b, t_0)$ corresponding to it. Using the same notation as in reference 5, we have

$$W_0(Q_0)W(Q, t; Q_0, t_0) = (2\pi)^{-2}\int_{-\infty}^{+\infty}\int \{\exp(i\xi Q + i\eta Q_0)\} Z(a, t; b, t_0)\, d\xi\, d\eta, \quad (1)$$

where

$$a = i\xi\Theta, \quad b = i\eta\Theta, \quad \Theta = kT.$$

The following relations follow from the definition of Z (see reference 5)

$$\overset{-a}{\dot{Q}} = -(\Theta/a)\dot{Z}/Z|_{b=0}, \quad (2)$$

$$\overset{-a}{Q^j} = (-\Theta)^j \frac{1}{Z}\frac{\partial^j Z}{\partial a^j}\bigg|_{b=0}, \quad (3)$$

where the index a denotes that we have taken an average over an auxiliary ensemble which is distinguished from the original, equilibrium one by the inclusion of an additional constant force $-a$, acting in the direction of the coordinate Q.

The equation of motion for Q averaged over the ensemble just mentioned connects $\overset{-a}{\dot{Q}}$ with the moments $\overset{-a}{Q^j}$ and, would according to (2) and (3) at the same time be a differential equation for the characteristic function Z. Using the inverse Fourier transformation one can from this equation go over to a differential equation for the probability density itself.

The transition probability density $W(Q, t; Q_0, t_0)$ is the solution of this equation which has a source for $t = t_0$.

2. A BROWNIAN PARTICLE IN AN EXTERNAL FIELD

The averaged equation of motion for a particle in the presence of an additional force $(-a)$ and

neglecting the inertial forces is of the form

$$\gamma\overline{\dot{Q}}^{a} + \overline{F(Q)}^{a} = -a, \tag{4}$$

where γ is the coefficient of viscosity and $F(Q) = -dU/dQ$ is the external force.

Expanding $F(Q)$ in a power series in Q and averaging (4) term by term we find

$$\gamma\overline{\dot{Q}}^{a} + \sum_j A_j \overline{Q^j}^{a} = -a,$$
$$A_j = (1/j!)\, d^{j+1}U/dQ^{j+1}|_{Q=0}. \tag{4'}$$

Substituting (2) and (3) into (4′) we find a differential equation for the characteristic function $Z(a,t)$:

$$\gamma\frac{\partial Z}{\partial t} - \frac{a}{\Theta}\sum_j A_j(-\Theta)^j\frac{\partial^j Z}{\partial a^j} = \frac{a^2}{\Theta}Z. \tag{5}$$

Evaluating the Fourier components of this equation, according to (1), we find that $W(Q,t;Q_0,t_0)$ satisfies the Einstein-Fokker-Planck equation

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial Q}\left(D\frac{\partial W}{\partial Q} - \frac{1}{\gamma}\frac{\partial U}{\partial Q}W\right), \qquad D = \frac{\Theta}{\gamma}, \tag{6}$$

being its source function. One sees easily that the inclusion of an additional constant force $-a$ leads formally to the appearance of a diffusion current on the right hand side of Eq. (6).

3. AN ELECTRICAL CIRCUIT WITH A NON-LINEAR RELAXATION MECHANISM

We shall consider an electrical circuit consisting of a capacity C and a resistance with a nonlinear current-voltage characteristic of the form

$$i = f(u) = \sum A_j u^j.$$

The charge Q concentrated upon the capacity is the generalized coordinate and the voltage u the generalized force so that the averaged equation of motion for Q when there is an additional constant force $-a$ present is of the form

$$\overline{\dot{Q}}^{a} + \overline{f[(Q/C)+a]}^{a} = 0. \tag{7}$$

By means of the procedure described in Sec. 2 we find the following differential equation for the characteristic function $Z(a,t)$:

$$\frac{\partial Z}{\partial t} = \frac{a}{\Theta}\sum_{j,k} A_j\binom{j}{k}(-\Theta/C)^k a^{j-k}\frac{\partial^k Z}{\partial a^k}. \tag{8}$$

The corresponding equation for the probability density itself

$$W(q,\tau) = (2\pi\Theta i)^{-1}\int_{-i\infty}^{+i\infty} e^{aQ/\Theta}Z(a,t)\,da$$

is of the form

$$\partial W/\partial\tau = \partial J/\partial q;$$
$$J = \sum_{j,k} B_j\binom{j}{k}\frac{\partial^k}{\partial q^k}(q^{j-k}W) = \sum_k \frac{1}{k!}\frac{\partial^k}{\partial q^k}\left(W\frac{d^kG}{dq^k}\right), \tag{9}$$

where we have introduced the notation

$$q = Q(C\Theta)^{-1/2}, \quad \tau = t(C\Theta)^{-1/2}, \quad B_j = A_j(\Theta/C)^{j/2},$$
$$G(q) = \sum_j B_j q^j = f(q\sqrt{\Theta/C}).$$

The solution of this equation can be obtained in the form of quadratures. The expression for the current J can be transformed to the form

$$J = \exp(-q^2/2)\,J_1, \qquad J_1 = \sum_{j,k}\frac{B_j j!}{2^k k!(j-2k)!}\frac{d^{j-2k}W_1}{dq^{j-2k}}, \tag{10}$$

$$W_1 = W\exp(q^2/2), \tag{10'}$$

after which (9) can be written as

$$\partial W_1/\partial\tau = \partial J_1/\partial q - qJ_1. \tag{11}$$

writing after that

$$W_1(q,\tau) = (2\pi i)^{-1}\int_{-i\infty}^{+i\infty} e^{qs}\overline{W}_1(s,\tau)\,ds, \tag{12}$$

we obtain for the Fourier component $\overline{W}_1$ the equation

$$\frac{\partial\overline{W}_1}{\partial\tau} = s\overline{J_1}(s)\overline{W}_1 + \frac{\partial}{\partial s}[\overline{J_1}(s)\overline{W}_1],$$

$$\overline{J_1}(s) = \sum_{j,k}\frac{B_j j!\, s^{j-2k}}{2^k k!(j-2k)!} = \sum_j \frac{B_j i^j}{2^{j/2}}H_j(s/i\sqrt{2})$$

$$= (2\pi)^{-1/2}\int_{\infty}^{+\infty} e^{-(q-s)^2/2}G(q)\,dq \tag{13}$$

[$H_n(x)$ is a Hermite polynomial] which after the substitution

$$\overline{W}_2 = \overline{J_1}\overline{W}_1\exp(s^2/2) \tag{14}$$

leads to

$$\partial\overline{W}_2/\partial\tau - \overline{J_1}\,\partial\overline{W}_2/\partial s = 0. \tag{15}$$

The initial condition $\overline{W}_2(s,\tau_0) = \overline{W}_{20}(s)$ is connected with the initial condition of the original equation (9) $W(q,\tau_0) = W_0(q)$ through (10′), (12), and (14).

Solving (15) by the methods of characteristics we find $\overline{W}_2(s,\tau) = \Phi(\psi)$ where $\psi(s,\tau) = \int ds/\overline{J}_1(s) + \tau$ is the characteristic of Eq. (15).

Taking the initial condition into account we get $\overline{W}_2(s,\tau) = \overline{W}_{20}[p(s,\tau)]$ while $p(s,\tau)$ is determined from the condition

$$\int_s^p d\xi/\overline{J_1}(\xi) = \tau - \tau_0.$$

If we now use (10′), (12), and (14) to go from $\overline{W}_2$ to W, we get

$$W(q,\tau)=(2\pi i)^{-1}\int_{-i\infty}^{+i\infty}\int_{-\infty}^{+\infty}\exp\left\{\tfrac{1}{2}\left[(q'-p)^2-(q-s)^2\right]\right\}$$

$$\times W_0(q')\frac{\overline{J_1}(p)}{\overline{J_1}(s)}\,ds\,dq'=(2\pi i)^{-1}$$

$$\times\int_{-i\infty}^{+i\infty}\int_{-\infty}^{+\infty}W_0(q')\exp\left\{\tfrac{1}{2}\left[(q'-p)^2-(q-s)^2\right]\right\}dp\,dq'. \quad (16)$$

Substituting $W_0(q')=\delta(q'-q_0)$ into (16) we find the source function of Eq. (9).

$$W(q,\tau;q_0,\tau_0)=(2\pi i)^{-1}\int_{-i\infty}^{+i\infty}\exp\left\{\tfrac{1}{2}\left[(q_0-p)^2-(q-s)^2\right]\right\}dp,$$

$$\int_{p}^{s(p,\tau)}d\xi/\overline{J_1}(\xi)+\tau-\tau_0=0,$$

$$\overline{J_1}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-(q-\xi)^2/2}G(q)\,dq, \quad (17)$$

which also gives the probability density for a transition of the circuit from a state Q_0 at time t_0 to a state Q at time t, if the current-voltage characteristic $i=f(q\sqrt{\Theta/C})=G(q)$ of the nonlinear resistance which enters into the circuit is known.

To study Eq. (17) further we perform a change of variables, putting

$$p-q_0=\xi(C\Theta)^{-1/2},$$

$$W(q,\tau;q_0,\tau_0)\,dq=W(Q,t;Q_0,t_0)\,dQ,$$

after which we get, shifting the contour over which we integrate,

$$W(Q,t;Q_0,t_0)=\frac{1}{2\pi iC\Theta}\int_{-i\infty}^{+i\infty}\exp\left\{\frac{1}{2C\Theta}\left[\xi^2-(Q-\eta(\xi,t))^2\right]\right\}d\xi,$$

$$\int_{Q_0+\xi}^{\eta(\xi,t)}d\rho/F_\Theta(\rho)+t-t_0=0,$$

$$F_\Theta(\rho)=\frac{1}{\sqrt{2\pi C\Theta}}\int_{-\infty}^{+\infty}\exp\left[-\frac{(Q'-\rho)^2}{2C\Theta}\right]F(Q')\,dQ',$$

$$F(Q')=f(Q'/C). \quad (17')$$

We shall investigate the behavior of W as $\Theta\to 0$. In that case

$$(2\pi C\Theta)^{-1/2}\exp(\xi^2/2C\Theta)\to\delta(x),$$

where $x=-i\xi$ so that

$$W(Q,t;Q_0,t_0)$$

$$\to(2\pi C\Theta)^{-1/2}\exp\{-[Q-Q_\Theta(Q_0,t)]^2/2C\Theta\}, \quad (18)$$

while $Q_\Theta(Q_0,t)$ is determined by the condition

$$\int_{Q_0}^{Q_\Theta}d\rho/F_\Theta(\rho)+t-t_0=0, \quad (19)$$

i.e., satisfies the differential equation

$$Q_\Theta+F_\Theta(Q_\Theta)=0. \quad (20)$$

If we also take into account that as $\Theta\to 0$, $F_\Theta(Q_\Theta)\to F(Q_\Theta)$, according to its definition, while (18) goes over into a δ-function, we get in the limit $\Theta=0$

$$W(Q,t;Q_0,t_0)=\delta[Q-Q(Q_0,t)], \quad (21)$$

where $Q(Q_0,t)$ is the solution of the original non-averaged equation

$$\dot{Q}+f(Q/C)=0 \quad (7')$$

a result which we should have expected.

If we now go over to the case $\Theta\neq 0$ we shall "spread out" the original δ-shaped distribution (21) over the neighborhood of the trajectory given by (7').

Putting $\eta(\xi,t)=Q_\Theta(Q_0,t)+\eta'(\xi,t)$ we find η' from the condition

$$\int_{Q_0+\xi}^{Q_\Theta+\eta'}d\rho/F_\Theta(\rho)+t-t_0=0. \quad (22)$$

Expanding (22) in the neighborhood of $\xi=0$ in powers of ξ, η' and using (19) we get

$$\eta'=\varepsilon(Q_0,t)\,\xi,$$

$$\varepsilon(Q_0,t)=F_\Theta(Q_\Theta)/F_\Theta(Q_0)=\dot{Q}_\Theta/\dot{Q}_0. \quad (23)$$

After that substituting (23) into (17), and evaluating the integral for W, we find finally

$$W(Q,t;Q_0,t_0)$$

$$=[2\pi\sigma(Q_0,t)]^{-1/2}\exp\{-[Q-Q_\Theta(Q_0,t)]^2/2\sigma(Q_0,t)\} \quad (24)$$

$$\sigma(Q_0,t)=C\Theta[1-\varepsilon^2(Q_0,t)]. \quad (25)$$

In the approximation under consideration the Brownian motion of a non-linear system is thus described by a Gaussian distribution for the transition probability while the dispersion of this distribution depends not only on the time, but also on the initial conditions, according to (23) and (25). The center of the distribution (the average value of the coordinate) is $Q_\Theta(Q_0,t)$.

The average value of the coordinate satisfies Eq. (25) which is in general different from the non-averaged Eq. (7'). It follows from the definition of $F_\Theta(Q)$ [see the third equation of (17)] that this difference is only absent for linear systems ($F_\Theta(Q)\sim Q$), and for nonlinear systems it vanishes only when there is no thermal noise ($\Theta=0$).

In conclusion I express my gratitude to Prof. Ya. P. Terletskiĭ for his interest in this paper.

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THE PHENOMENOLOGICAL THEORY OF THE VOIGT EFFECT IN PARAMAGNETICS

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A macroscopic calculation of the Voigt effect for centimeter waves in paramagnetic media is presented.

1. Ordinary double refraction in gyrotropic media denotes that radiation traversing such substances splits into two waves with mutually perpendicular magnetic (and electric) vectors and propagating in different directions. In the special case represented by the Voigt effect the gyration vector is perpendicular to the direction of propagation of the incident wave. The original linearly polarized radiation then splits into two waves which propagate in the same direction with different velocities and at different rates of absorption. These waves interfere at each point of the medium; the original linear polarization becomes elliptic and the ellipse rotates as the radiation passes through the medium.

In the present paper we attempt to construct a phenomenological theory of paramagnetic rotation through the Voigt effect for centimeter waves. The paramagnetic substance is assumed to be electrically isotropic with magnetic anisotropy caused by a static external magnetic field.[1]

2. We must first derive the magnetic susceptibility tensor for arbitrary relative orientation of the external static magnetic field $\mathbf{H}_0$ and the external oscillating magnetic field $\boldsymbol{\eta}$ of the radio wave. Hitherto[2,3] phenomenological theories of paramagnetic relaxation effects have considered only two special orientations, with $\mathbf{H}_0$ and $\boldsymbol{\eta}$ either mutually parallel or perpendicular.

To obtain the magnetic susceptibility tensor we use the general equations that describe the magnetization of a normal paramagnetic with pure spin magnetism in an oscillating field.[2] The equation for the time dependence of the magnetization $\mathbf{M}$ is[2]

$$\dot{\mathbf{M}} = -\varkappa\{\partial\Phi/\partial\mathbf{M}\} + g\{[\mathbf{M}\times\mathbf{H}]\}, \tag{1}$$

where

$$\Phi = -b/2T - \mathbf{HM} + \mathbf{M}^2T/2C \tag{2}$$

is the nonequilibrium thermodynamic potential. Here $\mathbf{H} = \mathbf{H}_0 + \boldsymbol{\eta}$ is the total external magnetic field with a dc component $\mathbf{H}_0$ and an rf component $\boldsymbol{\eta}$; T is the temperature of the spin system; b is the magnetic specific heat constant; C is the Curie constant; g is the gyromagnetic ratio; $\varkappa$ in the phenomenological theory is an unknown function of H_0 and of the (assumed constant) lattice temperature T_0. The curly brackets in (1) denote that the quantities within them are linearized with respect to the small quantities $\theta = T - T_0$, the components of $\boldsymbol{\eta}$ and the variable part of the magnetization $\boldsymbol{\xi} \equiv \mathbf{M} - \mathbf{M}_0$, where $\mathbf{M}_0 \equiv (C/T_0)\mathbf{H}_0$.

We use (1) and the first law of thermodynamics for the spin system, as in reference 2. Instead of $\varkappa$ we introduce the isothermal spin relaxation time of the magnetization, $\tau_S = c/T_0\varkappa$;[4] instead of the coefficient of thermal conductivity between the spin system and the lattice, α, which appears in the first law and remains an unknown function of T_0 and H_0 in the phenomenological theory, we introduce the spin-lattice relaxation time $\tau_e = (b + cH_0^2)/\alpha T_0$.[3] After linearization we arrive at the following basic equation for ξ:

$$(i\omega + \tau_s^{-1})\boldsymbol{\xi} + \frac{i\omega(1-\gamma)}{(i\omega\gamma + \tau_e^{-1})\tau_s}(\mathbf{l}_0\boldsymbol{\xi})\mathbf{l}_0 + \omega_0[\mathbf{l}_0\times\boldsymbol{\xi}]$$

$$= \chi_0\tau_s^{-1}\boldsymbol{\eta} + \chi_0\omega_0[\mathbf{l}_0\times\boldsymbol{\eta}], \tag{3}$$

where $\mathbf{l}_0$ is a unit vector in the direction of the constant field $\mathbf{H}_0$; $\chi_0 = c/T_0$ is the equilibrium isothermal magnetic susceptibility; $\gamma = [1 + H_0^2(c/b)]^{-1} = C_M/C_H$ is the ratio of specific heats of the spin system for constant magnetization and a static field; ω is the oscillating field frequency (in deriving (3) it was assumed that the time dependence of $\boldsymbol{\xi}$ and θ is given, as for $\boldsymbol{\eta}$, by the factor $\exp(i\omega t)$, since steady conditions are being considered) and $\omega_0 = gH_0$. It is found easily and directly that the solution of (3) is given by

$$\boldsymbol{\xi} = \chi_\perp\boldsymbol{\eta} + i[\boldsymbol{\delta}\times\boldsymbol{\eta}] + (\chi_\parallel - \chi_\perp)(\mathbf{l}_0\boldsymbol{\eta})\mathbf{l}_0. \tag{4}$$

Here $\chi_\parallel$ and $\chi_\perp$ represent the susceptibility of

the paramagnetic in parallel and perpendicular fields, respectively (obtained in reference 2) and δ is the gyration vector (obtained in reference 1), which is parallel to l_0. Substituting (4) into (3) we obtain the complex magnetic susceptibility tensor

$$\chi = \begin{pmatrix} \chi_\perp & -i\delta & 0 \\ i\delta & \chi_\perp & 0 \\ 0 & 0 & \chi_\parallel \end{pmatrix}. \tag{5}$$

3. In order to obtain the refractive index of a wave traversing a paramagnet we now make use of Maxwell's equations

$$\operatorname{curl} \mathbf{E} = -\dot{\mathbf{B}}/c, \qquad \operatorname{curl} \mathbf{H}' = \dot{\mathbf{D}}/c. \tag{6}$$

Here $\mathbf{H}'$ is the magnetic field in the material. Since the specimen is regarded as electrically isotropic we have

$$\mathbf{D} = \varepsilon \mathbf{E}. \tag{7}$$

We denote by η' and $\mathbf{b}$ the variable parts of $\mathbf{H}'$ and $\mathbf{B}$, so that $\mathbf{b} = \eta' + 4\pi\xi$. In paragraph 2 it was shown that $\xi = \chi\eta$; since paramagnets are only weakly polarizable we shall assume that $\xi = \chi\eta'$. Equation (4) then gives

$$\mathbf{b} = (1 + 4\pi\chi_\perp)\,\eta' + 4\pi\,[\delta \times \eta'] + 4\pi\,(\chi_\parallel - \chi_\perp)\,(l_0\eta')\,l_0. \tag{8}$$

Substituting (7) into (6), eliminating $\mathbf{E}$ and making use of (8), we obtain the wave equation for η', which for a plane wave is represented by

$$(\varepsilon n^2)^{-1}\,[k_0\,(k_0\eta') - \eta'] + (1 + 4\pi\chi_\perp)\,\eta' + 4\pi i\,[\delta \times \eta'] + 4\pi\,(\chi_\parallel - \chi_\perp)\,(l_0\eta')\,l_0 = 0, \tag{9}$$

where n is the index of refraction and k_0 is a unit vector in the direction of wave propagation.

We now consider a wave propagating in the x direction. Projecting (9) on the coordinate axes, we obtain a set of homogeneous equations for the vector components of η', the solution of which gives two values for the index of refraction. Thus two waves are possible in the x direction. From the solution we find that one of these waves possesses the index of refraction

$$n_\parallel = \sqrt{\varepsilon(1 + 4\pi\chi_\parallel)} \tag{10}$$

and a magnetic vector parallel to l_0 (so that $\eta'_X = n'_y = 0$, $n'_Z \neq 0$). For the other wave we have the refractive index

$$n_\perp = \{\varepsilon\,[(1 + 4\pi\chi_\perp) - 16\pi^2\delta^2/(1 + 4\pi\chi_\perp)]\}^{1/2} \tag{11}$$

and a magnetic vector lying in a plane perpendicular to l_0 (so that $\eta_Z = 0$). The second wave is elliptically polarized with the following relation between η'_X and η'_y:

$$\eta_x'^{(0)} = i4\pi\delta\eta_y'^{(0)}/(1 + 4\pi\chi'_\perp), \tag{12}$$

which shows that the x component of η' is considerably smaller than the y component, since $\chi_\perp$ and δ are of the order of $\chi_0 \sim 10^{-6}$ (see reference 1).

4. Let us now consider the interference of the waves indicated by (10) and (11) inside the paramagnetic. An rf wave impinges at the point $x = 0$ on a plane-parallel paramagnetic plate which is perpendicular to the x axis. The magnetic field of the wave is linearly polarized in the yz plane with the components

$$\eta_z = (\eta_0 \cos\alpha)\,e^{i\omega t}, \qquad \eta_y = (\eta_0 \sin\alpha)\,e^{i\omega t}, \tag{13}$$

where α is the angle between $\mathbf{H}_0$ and η. Taking the boundary conditions into account and ignoring reflection of the incident wave (since we are interested only in the change of polarization but not of intensity), we obtain at a point x inside the plate the two waves:

$$\eta'_z = \eta_0 \cos\alpha \exp\{i\omega\,(t - n_\parallel x/c)\}, \tag{14}$$

$$\eta'_y = \eta_0 \sin\alpha \exp\{i\omega\,(t - n_\perp x/c)\},$$

$$\eta'_x = i\,\frac{4\pi\delta}{1 + 4\pi\chi_\perp}\,\eta'_y, \tag{15}$$

the complex refractive indices of which are given by

$$n_\parallel = n'_\parallel - in''_\parallel, \qquad n_\perp = n'_\perp - in''_\perp. \tag{16}$$

To investigate the interference of waves (14) and (15) it is sufficient to obtain the results in the yz plane, since the x component of the magnetic field disappears when the combined oscillations emerge from the plate. Introducing the notation

$$a_1 = \eta_0 \sin\alpha \exp\left(-\frac{\omega}{c}\,n''_\perp x\right), \qquad \delta_1 = -\frac{\omega}{c}\,n'_\perp x,$$

$$a_2 = \eta_0 \cos\alpha \exp\left(-\frac{\omega}{c}\,n''_\parallel x\right), \qquad \delta_2 = -\frac{\omega}{c}\,n'_\parallel x \tag{17}$$

and eliminating the time t from η'_y and η'_z, we obtain the equation of an ellipse:

$$(\eta'_y/a_1)^2 + (\eta'_z/a_2)^2 - 2\,(\eta'_y\eta'_z/a_1a_2)\cos\delta = \sin^2\delta, \tag{18}$$

where $\delta = \delta_2 - \delta_1$. This ellipse is rotated with respect to the coordinate axes through an angle which is the sum of α and the angle of paramagnetic rotation β, by which we mean the angle between η in the incident wave and the major semi axis of the polarization ellipse of the emerging wave. The ellipse can be put into canonical form by a transformation of the axes, and for the angle of rotation $\varphi = \alpha + \beta$ we obtain

$$\tan 2\varphi = 2a_1a_2\cos\delta/(a_2^2 - a_1^2); \tag{19}$$

it is evident that we have $\varphi = \alpha$ at $x = 0$, as is to

be expected. We now take the very small susceptibility of paramagnets ($\chi_0 \sim 10^{-6}$) into account. Since $\chi_\perp$, $\chi_\parallel$ and δ in (10) and (11) are of the order of χ_0 (see reference 1), we can extract approximate values of the square roots in (10) and (11) and obtain corresponding approximations for $\cos\delta$ and for the exponentials in a_1 and a_2. Equation (19) then gives

$$\alpha + \beta = \tan^{-1}\left\{\frac{\sin\alpha}{\cos\alpha}\left[1 - \frac{\omega}{c}(n''_\perp - n''_\parallel)x\right]\right\}. \quad (20)$$

Considering also that the investigated effect is very small in paramagnets, in virtue of which we may assume $\sin\beta \sim \beta$ and $\cos\beta \sim 1$, we finally obtain from (20):

$$\beta = -(\pi\sqrt{\varepsilon}\omega/c)(\chi''_\perp - \chi''_\parallel)(\sin 2\alpha)x, \quad (21)$$

where χ'' is the imaginary part of the susceptibility.

It is evident from (21) that the angle of rotation depends in first approximation only on the difference between the absorptions of the waves corresponding to (10) and (11). This angle also exhibits periodic dependence on α, vanishing at $\alpha = 0$, $\pi/2$, π . . . . The angle of rotation also depends on the frequency of the oscillating field and the magnitude of the dc field.

5. In experiments on paramagnetic rotation it is customary to determine the dependence of the angle of rotation on the static field for a constant radiation frequency. To plot the $\beta(H_0)$ curve we must use the expressions[2]

$$\chi''_\perp = \chi_0 \frac{[1 + \tau_s^2(\omega_0^2 + \omega^2)]\tau_s\omega}{[1 + \tau_s^2(\omega_0^2 - \omega^2)]^2 + 4\tau_s^2\omega^2}, \quad (22)$$

$$\chi''_\parallel = \chi_0 \frac{(F\tau_e + \tau_s)\omega + (1-F)^2\tau_e^2\tau_s\omega^3}{[1 - (1-F)\tau_e\tau_s\omega^2]^2 + (\tau_e + \tau_s)^2\omega^2}, \quad (23)$$

where $F = 1 - \gamma$ increases monotonically from 0 to 1 as H_0 increases.

All the rotation experiments known to us were performed at frequencies for which $\tau\omega$ was very close to unity and $\tau_e\omega$ was so large that spin-lattice relaxation was practically absent. For $\tau_e\omega \gg 1$, Eq. (23) then gives

$$\chi''_\parallel = \chi_0(1-F)^2\tau_s\omega/[1 + (1-F)^2\tau_s^2\omega^2]. \quad (24)$$

It is evident from (22) and (24) that the static field H_0 enters into the expression for β both directly through ω_0 and F and through the isothermal spin relaxation time τ_S. For some paramagnets experimental data have recently been obtained[5] on spin absorption in parallel fields which may possibly indicate that τ_S is dependent on the external field (although this is still unaccounted for) or perhaps that the theory of reference 2 is inadequate for these cases. However, a number of investigations[1,6] show that the theory of Shaposhnikov[2] is applicable with τ_S independent of H_0 for a large number of paramagnets over broad frequency and temperature ranges.

When this is so, Eqs. (21), (22), and (23) give a very definite dependence of β on H_0. The shape of the $\beta(H_0)$ curve can, of course, differ depending on the substance. In some substances $\chi''_\parallel$ is very weakly dependent on H_0;[6] the curve $\beta = f(H_0)$ is then similar in shape to the absorption curve in perpendicular fields (22), with the single difference that since $\chi''_\perp$ and $\chi''_\parallel$ are equal at $H_0 = 0$ the rotation curve starts at the coordinate origin (which is physically expected, since with $H_0 = 0$ there is no gyrotropy and there can be no rotation). We also note from an analysis of the rotation curve given by (21), (22), and (24) with τ_S independent of H_0 that except for $H_0 \to \infty$ and $H_0 = 0$ the rotation either vanishes nowhere or vanishes twice.

6. Not much experimental information has been published concerning the effect that we are considering. So far as we know the first such data were given in reference 7 for powdered $MnSO_4 \cdot H_2O$ at about 9.4×10^9 cps at room temperature. In this paper the experimental curve $\beta = f(H_0)$ was given for H_0 ranging from about 2.7×10^3 to 4.7×10^3 oersteds, with a maximum for H_0 at about 3.4×10^3 oersteds and with no intersection of the horizontal axis. The review article by Gozzini[3] gives the curve in reference 7 together with a curve for $MnSO_4 \cdot H_2O$ (communicated privately) obtained under the same conditions but carried to about 5.8×10^3 oersteds for H_0, as well as measurements of the given effect in certain radicals. The curves given in reference 8 possess a single maximum but do not intersect the horizontal axis. Hedvig[3] gives experimental results for the dependence of the rotation angle on the angle α between the static and oscillating fields for the organic free radical diphenylpicrylhydrazyl at about 9.4×10^9 cps at room temperature; this relation can apparently be characterized by $\sin 2\alpha$. Battaglia et al and Hedvig note especially that the sign of the effect does not change. In these papers the experimental results are not discussed from a theoretical point of view.

A recent paper by Imamutdinov, Neprimerov and Shekun[10] gives the experimental $\beta(H_0)$ curve for powdered $MnCl_2 \cdot 4H_2O$ at room temperature at about 9.4×10^9 cps with some theoretical discussion.

The rotation curve given in reference 10 begins

at zero, has a sharp maximum in the positive region, passes through zero again at $H_0 = 4 \times 10^3$ oersteds, has a very indistinct minimum in the negative region and then monotonically approaches the horizontal axis slowly from below. The theoretical discussion is as follows. In (21) (given in reference 10 without derivation) the expressions

$$\chi''_{\perp} = \frac{\chi_0 \omega}{2\tau}\left[\frac{1}{(\omega_0-\omega)^2+\tau^{-2}} + \frac{1}{(\omega_0+\omega)^2+\tau^{-2}}\right], \quad (25)$$

$$\chi''_{\parallel} = (\chi_0 \omega/\tau)/(\omega^2+\tau^{-2}), \quad (26)$$

are substituted. It is stated[10] that for lack of a satisfactory theory of the complex paramagnetic susceptibility of solids these expressions were obtained by using the theory for a paramagnetic gas consisting of identical particles with spin $\frac{1}{2}$. It is then stated that if we assume the relaxation time τ in (25) and (26) to increase with H_0 these equations "describe the experimental results well both qualitatively and quantitatively." Regarding the postulated increase of τ with the field the writers refer to Gorter's book[3] and Kurushin's papers.[6] It is also stated that the phenomenological theory of Shaposhnikov[3] likewise apparently leads to (25) and (26).

7. A comparison of the experimental data from references 7 — 9 (in Sec. 6) with the discussion of the theoretical rotation curve given by (21), (22), and (24) (Sec. 5) indicates agreement between these experimental results and our theory. The following must be stated concerning the experimental rotation curve given in reference 10. For powdered $MnCl_2 \cdot 4H_2O$ at room temperature and at a frequency very close to that used in reference 7, Kurushin obtained the experimental curves of $\chi''_{\perp}(H_0)$ and $\chi''_{\parallel}(H_0)$ using the same apparatus that he had described in an earlier paper.[6] These results, which will be published in the near future, show without any doubt that up to H_0 of the order 6×10^3 oersteds the $\chi''_{\parallel}(H_0)$ curve is everywhere below the $\chi''_{\perp}(H_0)$ curve. It follows (see (21)) that in the given region of field values the rotation curve $\beta(H_0)$ can become zero nowhere except at the origin $H_0 = 0$; however, the experimental curve in reference 7 passes through zero for $H_0 = 4 \times 10^3$ oersteds. It must be added that for $MnCl_2 \cdot 4H_2O$ under the given conditions many experiments[1,6] support Shaposhnikov's theory with τ_S independent of H_0; but in this case, as was noted at the end of Sec. 5, an analysis of the $\beta(H_0)$ curve shows that except for $H_0 = 0$ the curve has either no zero or two zeros, in agreement with the experimental results of Kurushin that have just been mentioned.

With regard to the theoretical discussion in reference 10 we may state the following. Although (25) and (26) are actually derived from the general equations of Karplus and Schwinger[11] for the complex susceptibility of an ideal gas in the special case of spin $\frac{1}{2}$ and perpendicular and parallel fields, we believe it would be incorrect to use (25) and (26) in the theoretical explanation of the experimental rotation curve given in reference 10. Under the given experimental conditions the spin system of the paramagnetic is practically isolated from the lattice, since $\tau_e\omega \gg 1$. Therefore for $\chi''_{\parallel}$ we must use an expression that corresponds to adiabatic spin-spin relaxation in the absence of spin-lattice relaxation, as we did in Sec. 5 [see Eq. (24)]. It is also evident from (22) that $\chi''_{\perp}$ does not contain τ_e at all. A comparison of (25) and (26) with (22) and (24) shows that when $\tau = \tau_S$ (25) coincides with (22); however, (26) is not transformed into (24), which corresponds to adiabatic spin-spin relaxation and is obtained from (23) for $\tau_e\omega \to \infty$, but (26) does correspond to isothermal spin-spin relaxation and is obtained from (23) for $\tau_e\omega \to 0$. We also note that if we follow the authors of reference 10 by accepting (25) and (26), we cannot fully check their statement that theory and experiment are in good qualitative and quantitative agreement for τ increasing with H_0 since the specific dependence of τ_0 on H is not given. We must add that if in (25) and (26) we understand τ to mean τ_S [which is required if (25) is to agree with (22)], then we are not in accord with the reference to Gorter and to Kurushin[6] with regard to the dependence of τ_S on H_0. In our opinion Gorter's results cannot be used for any definite general conclusions regarding the dependence of τ_S on H_0, while the cited papers of Kurushin indicate that τ_S is independent of H_0.

8. We have plotted the $\beta(H_0)$ curve given by (21), (22), and (24) for $MnCl_2 \cdot 4H_2O$ under the experimental conditions of reference 10 with τ_S taken as 0.24×10^{-9} sec (see reference 6). The curve starts at zero and remains entirely in the positive region with one maximum, after which it drops monotonically and practically reaches the horizontal axis for H_0 of the order 8×10^3 oersteds. This rotation curve agrees with the experimental curves of $\chi''_{\perp}(H_0)$ and $\chi''_{\parallel}(H_0)$ that were obtained by Kurushin (see Sec. 7).

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NONRELATIVISTIC SOLUTION OF THE BETHE-SALPETER EQUATION

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A nonrelativistic solution of the Bethe-Salpeter equation has been obtained which includes terms of the order of the relative velocity of the particles.

THE Bethe-Salpeter equation for two identical Dirac particles with mass m and charge e has the following form in momentum space:

$$\left(\tfrac{1}{2}\hat{K}^a+\hat{p}^a-m\right)\left(\tfrac{1}{2}\hat{K}^b-\hat{p}^b-m\right)\psi(p) = \frac{ie^2}{4\pi^3}\int\frac{\gamma_\nu^a\gamma_\nu^b}{k^2}\psi(p-k)\,d^4k, \quad (1)$$

where K is the total, and p the relative, momentum of the system. We always set $\hbar = c = 1$, and the scalar product of any two vectors q and τ is taken as $q\tau = q_\nu\tau_\nu = q_0\tau_0 - q_1\tau_1 - q_2\tau_2 - q_3\tau_3$. The index a refers to the Dirac matrices belonging to the first particle, and the index b, to those belonging to the second particle. We have $\gamma_0 = \beta$, $\gamma_{1,2,3} = \beta\alpha_{1,2,3}$ and $\hat{q} = q_\nu\gamma_\nu$. Finally,

$$(\hat{q}^a\psi)_{\rho_1\rho_2} = (\hat{q}^a)_{\rho_1\rho_1'}\psi_{\rho_1'\rho_2}, \qquad (\hat{q}^b\psi)_{\rho_1\rho_2} = (\hat{q}^b)_{\rho_2\rho_2'}\psi_{\rho_1\rho_2'} = \psi_{\rho_1\rho_2'}(\hat{q}^{bT})_{\rho_2'\rho_2}.$$

We multiply both sides of (1) by the operator

$$\left(\tfrac{1}{2}\hat{K}^a+\hat{p}^a+m\right)\left(\tfrac{1}{2}\hat{K}^b-\hat{p}^b+m\right),$$

and obtain in the coordinate system in which the two-particle system as a whole is at rest ($\mathbf{K} = 0$, $K_0 = 2m - E$, where E is the binding energy)

$$\left[\left(m-\frac{E}{2}+p_0\right)^2-(\mathbf{p}^2+m^2)\right]\times\left[\left(m-\frac{E}{2}-p_0\right)^2-(\mathbf{p}^2+m^2)\right]\psi(p) = \frac{ie^2}{4\pi^3}\left[(1+\beta^a)m-\frac{E}{2}+p_0+(\alpha^a\mathbf{p})\right]\times\left[(1+\beta^b)m-\frac{E}{2}-p_0-(\alpha^b\mathbf{p})\right]\times\int\frac{1-(\boldsymbol{\alpha}^a\boldsymbol{\alpha}^b)}{k_0^2-\mathbf{k}^2}\psi(p-k)\,d^4k. \quad (2)$$

In the nonrelativistic case, $E/m \ll 1$, and the wave function $\psi(p)$ differs from zero appreciably in the region of small momenta, $|\mathbf{p}/m| = v \ll 1$; where v is the relative velocity of the particles, and of small p_0 (the magnitude of p_0 is of the same order as E). In the following we shall be interested in the solution of equation (2) with an accuracy up to and including terms of order v. On the right hand side of (2) we can therefore neglect the terms k_0^2 and $(\boldsymbol{\alpha}^a\boldsymbol{\alpha}^b)$ under the integral, as this neglection is equivalent to replacing the retarded interaction by the instantaneous Coulomb interaction,[1] which is an effect of higher order [the retardation effect gives rise to terms which are smaller than the Coulomb potential by the factor v^2 (reference 2)]. Discarding all terms of higher order of smallness, we rewrite Eq. (2) in the form

$$\left(p_0-\frac{E}{2}-\frac{\mathbf{p}^2}{2m}\right)\left(p_0+\frac{E}{2}+\frac{\mathbf{p}^2}{2m}\right)\psi(\mathbf{p},\,p_0) = \frac{ie^2}{2\pi^2}\left[\frac{1}{2}(1+\beta^a)+\frac{(\alpha^a\mathbf{p})}{2m}\right]\left[\frac{1}{2}(1+\beta^b)-\frac{(\alpha^b\mathbf{p})}{2m}\right]\times\int\frac{1}{\mathbf{k}^2}\psi(\mathbf{p}-\mathbf{k},\,t=0)\,d^3k$$

where $\psi(p) = \psi(\mathbf{p}, p_0)$. We also used the relation

$$\int\psi(\mathbf{p},\,p_0)\,dp_0 = 2\pi\psi(\mathbf{p},\,t=0).$$

Multiplying both sides of Eq. (3) by the expression

$$\frac{\exp(-ip_0t)}{2\pi(p_0-E/2-\mathbf{p}^2/2m)(p_0+E/2+\mathbf{p}^2/2m)} \quad (4)$$

and integrating over p_0 under the assumption that m in (1) has an infinitesimal negative imaginary part (this gives rise to the infinitesimal terms $+i\delta$ and $-i\delta$, where $\delta > 0$, in the first and second parentheses of the denominator of (4), respectively), we obtain, for $t > 0$,

$$\psi(\mathbf{p},\,t) = \frac{e^2}{2\pi^2}\frac{\exp\{-i(E/2+\mathbf{p}^2/2m)t\}}{E+p^2/m}\left[\frac{1}{2}(1+\beta^a)+\frac{(\alpha^a\mathbf{p})}{2m}\right]\times\left[\frac{1}{2}(1+\beta^b)-\frac{(\alpha^b\mathbf{p})}{2m}\right]\int\frac{1}{\mathbf{k}^2}\psi(\mathbf{p}-\mathbf{k},\,t=0)\,d^3k. \quad (5)$$

For the following it is convenient to introduce the notation

$$\psi(p) = \begin{pmatrix}\psi^\sigma(p) & \psi^{\mu_2}(p)\\ \psi^{\mu_1}(p) & \psi^{\mu_3}(p)\end{pmatrix}, \quad (6)$$

where $\psi^\sigma(\mathbf{p})$, $\psi^{\mu 1}(\mathbf{p})$, $\psi^{\mu 2}(\mathbf{p})$, and $\psi^{\mu 3}(\mathbf{p})$ are

two-rowed functions. Furthermore we set

$$\psi(\mathbf{p}, t=0) = \varphi(\mathbf{p}). \tag{7}$$

Using (5), (6), and (7), we find the following equations for the determination of the components of the wave function (7):

$$\left(E + \frac{\mathbf{p}^2}{m}\right)\varphi^{\sigma}(\mathbf{p}) = \frac{e^2}{2\pi^2}\int \frac{1}{\mathbf{k}^2}\left[\varphi^{\sigma}(\mathbf{p}-\mathbf{k}) + \frac{(\sigma^a \mathbf{p})}{2m}\varphi^{\mu_1}(\mathbf{p}-\mathbf{k}) - \frac{(\sigma^b \mathbf{p})}{2m}\varphi^{\mu_2}(\mathbf{p}-\mathbf{k}) - \frac{(\sigma^a \mathbf{p})(\sigma^b \mathbf{p})}{4m^2}\varphi^{\mu_3}(\mathbf{p}-\mathbf{k})\right] d^3k, \tag{8}$$

$$\left(E + \frac{\mathbf{p}^2}{m}\right)\varphi^{\mu_1}(\mathbf{p}) = \frac{e^2}{2\pi^2}\int \frac{1}{\mathbf{k}^2}\left[\frac{(\sigma^a \mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p}-\mathbf{k}) - \frac{(\sigma^a \mathbf{p})(\sigma^b \mathbf{p})}{4m^2}\varphi^{\mu_2}(\mathbf{p}-\mathbf{k})\right] d^3k, \tag{9}$$

$$\left(E + \frac{\mathbf{p}^2}{m}\right)\varphi^{\mu_2}(\mathbf{p}) = -\frac{e^2}{2\pi^2}\int \frac{1}{\mathbf{k}^2}\left[\frac{(\sigma^b \mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p}-\mathbf{k}) + \frac{(\sigma^a \mathbf{p})(\sigma^b \mathbf{p})}{4m^2}\varphi^{\mu_1}(\mathbf{p}-\mathbf{k})\right] d^3k, \tag{10}$$

$$\left(E + \frac{\mathbf{p}^2}{m}\right)\varphi^{\mu_3}(\mathbf{p}) = -\frac{e^2}{2\pi^2}\int \frac{1}{\mathbf{k}^2}\frac{(\sigma^a \mathbf{p})(\sigma^b \mathbf{p})}{4m^2}\varphi^{\sigma}(\mathbf{p}-\mathbf{k})\, d^3k. \tag{11}$$

It is seen from equations (8) to (11) that $\varphi^{\mu 1}(\mathbf{p})$ and $\varphi^{\mu 2}(\mathbf{p})$ are of the order of the relative velocity v, and the function $\varphi^{\mu 3}(\mathbf{p})$ is of the order v^2. In contrast to this, the function $\varphi^{\sigma}(\mathbf{p})$ contains "large components" which do not reduce to zero for $v = |\mathbf{p}/m| \to 0$. Since we are interested in the wave function (7) with an accuracy up to and including terms of order v, we shall discard the terms containing $\varphi^{\mu 1}$ and $\varphi^{\mu 2}$ on the right hand sides of Eqs. (8), (9), and (10), and set the function $\varphi^{\mu 3}$ equal to zero, as it is of order v^2. Furthermore, we replace the integral over the variable $\mathbf{k}$ Eqs. (9) and (10) by $(E + \mathbf{p}^2/m)\varphi^{\sigma}(\mathbf{p})$, using Eq. (8). As a result we find that the large components of the wave function (7) satisfy the Schrödinger equation, which in $\mathbf{x}$ space has the usual form*

$$\left(\frac{1}{m}\nabla^2 + \frac{e^2}{|\mathbf{x}|}\right)\varphi^{\sigma}(\mathbf{x}) = E\varphi^{\sigma}(\mathbf{x}), \tag{12}$$

The small components are given by

$$\varphi^{\mu_1}(\mathbf{x}) = \frac{(\sigma^a \nabla)}{2mi}\varphi^{\sigma}(\mathbf{x}), \qquad \varphi^{\mu_2}(\mathbf{x}) = -\frac{(\sigma^b \nabla)}{2mi}\varphi^{\sigma}(\mathbf{x}),$$
$$\varphi^{\mu_3}(\mathbf{x}) = 0. \tag{13}$$

In the mixed representation $\psi(\mathbf{p}, t)$, obtained from $\psi(\mathbf{x}, t)$ by a Fourier transformation with respect to the spatial coordinates $\mathbf{x}$ only, the wave function corresponding to the Bethe-Salpeter equation (1) has, according to (5), the following form for $t > 0$ (with an accuracy up to and including terms of order v):

$$\psi(\mathbf{p}, t) = \begin{pmatrix} \varphi^{\sigma}(\mathbf{p}) & -\frac{(\sigma^b \mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p}) \\ \frac{(\sigma^a \mathbf{p})}{2m}\varphi^{\sigma}(\mathbf{p}) & 0 \end{pmatrix} \exp\left\{-i\left(\frac{E}{2} + \frac{\mathbf{p}^2}{2m}\right)t\right\} \tag{14}$$

For $t < 0$ the exponent in (14) takes the opposite sign. It must be emphasized that the two-rowed (four-component) function $\varphi^{\sigma}(\mathbf{p})$ in (14) satisfies the Schrödinger equation and is also an eigenfunction of the operators of the total angular momentum of the system and its projection. This means that the function $\varphi^{\sigma}(\mathbf{p})$ satisfies, in the spinor indices, the corresponding operator equations of nonrelativistic quantum mechanics. Expression (14) is also valid for the mesonic interaction of the nucleons; however, in this case the Coulomb potential in the Schrödinger equation (12) must be replaced by the Yukawa potential or its generalization.

With the wave function (14) we can calculate the annihilation probability for bound particles (positronium) in the P state.[3,4] With the help of formula (14) it is possible to take account of the Coulomb interaction between the electron and the positron in the pair production by photons, such that the expression for the differential cross section contains all terms of the order of the relative velocity between the pair of particles. If Eq. (1) is to be solved with an accuracy up to and including terms of order v^2, the expression for the interaction energy between the electron and the positron will differ from the known result[5] by the absence of the exchange interaction, since Eq. (1) does not include exchange forces. To obtain the correct result, Eq. (1) has to be replaced by an equation of the Bethe-Salpeter type which contains, together with the ordinary interaction, a specific exchange interaction between the electron and the positron arising from their virtual annihilation.[6]

*Equation (12) was obtained by Bethe and Salpeter[1] as the nonrelativistic limit of Eq. (1).

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RESONANCE SCATTERING OF LOW-ENERGY GAMMA RAYS ON NUCLEI

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The mechanism of resonance scattering of low-energy gamma rays is discussed. It is shown that this scattering is of the nature of nuclear resonance fluorescence and that excitation of the nucleus can be described by means of one-nucleon transitions. The gamma-ray scattering cross section is calculated on the basis of the shell theory. Excited level widths are estimated from the Fermi gas model and the results obtained by Signell and Marshak[1] in connection with the theory of nucleon scattering. The results are in satisfactory agreement with the experimental data.

THE scattering of gamma rays with energies $E_\gamma < 30$ Mev exhibits two maxima, one of which is in the giant resonance region and is accounted for by the dipolar vibrations which can be excited in all nuclear matter. The other maximum* is below the particle threshold; Fuller and Hayward[3] have obtained experimental evidence that gamma-ray scattering in this energy region depends essentially on nuclear structure. The scattering cross section at the giant-resonance maximum increases relatively smoothly with the atomic number A, whereas the "subthreshold" scattering maximum varies sharply depending on the nucleus. The variation is especially great in the case of nuclei with a closed shell structure. It has also been found that scattering in this region exhibits a few very sharp resonances.

It was the purpose of the present work to determine the mechanism of gamma-ray scattering in the subthreshold region. It was assumed that scattering occurs at separate one-nucleon levels and can be described as nuclear resonance fluorescence. In the dipole approximation the cross section for gamma-ray scattering on a nucleus is given by

$$\sigma(\gamma, \gamma) = \frac{2\pi}{3}\frac{e^4\omega^2\hbar^2}{c^4M^2}\left|\sum_{n'}\frac{f_{nn'}}{E_\gamma - (E_{n'} - E_n) + i\Gamma/2)}\right|^2. \quad (1)$$

Here ω is the frequency of the scattered gamma ray; M is the nucleon mass; Γ is the total width of the intermediate (excited) level; $f_{nn'}$ is the oscillator strength for a transition from state n to state n′ and is defined (when the gamma ray is polarized along the Z axis of the nucleus) by

$$f_{nn'} = 2M\hbar^{-2}(E_{n'} - E_n)\,|Z_{nn'}|^2.$$

We shall consider real transitions corresponding to resonances. In the given approximation and for $E < E_{thr}$ as a rule one transition corresponds to each fixed ground level. Therefore only one term in the sum on the right side of (1) is important.

Schröder's method[4] was used to calculate the ground and excited nucleon levels. Although the calculation took strongly complicating factors into account — the Coulomb interaction of protons within the nucleus, the spin-orbit interaction, the diffuse nuclear boundary — this method permits a relatively simple and clear solution of the problem. On the other hand, consideration of these factors results in a more accurate charge distribution within the nucleus and in potentials that are in agreement with the phenomenological model of Feshbach, Porter, and Weisskopf.[5]

In the assumed model the radial part of the wave equation for neutrons is given by

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{2M}{\hbar^2}[E_{n,l,j} - V]\right\}R_{n,l,j}(r) = 0,$$

where

$$V = V_N - 0.38\left(\frac{\hbar}{m_\pi c}\right)^2\frac{1}{r}\frac{d}{dr}V_N\frac{(lS)}{\hbar^2}. \quad (2)$$

In (2) V_N is the potential for neutrons, while the second term represents the spin-orbit interaction. The specific character of V_N for Pb^{208} is given in reference 4.

A Coulomb interaction term must, of course, be added to the equation in order to determine the proton levels; in this procedure Schröder was followed exactly.

The total potential $V(r) + (\hbar^2/2M)\,l(l+1)\,r^{-2}$ was approximated by the parabola

*Bethe[2] has given a rough qualitative explanation of this resonance based on competition between reactions.

$$W = -u\left[1 - \left(\frac{r - r_1}{w_1}\right)^2\right].$$

The parameters u, r_1, w_1 are clearly functions of l and j. The eigenfunctions $E_{n,l,j}$ are given by

$$E_{n,l,j} = -u + (\hbar/w_1)\sqrt{2u/M}\,(n - 1/2),$$

and the solution of the radial part of the wave function is given by

$$R_{n,l,j}(r) = Nr^{-1}H_n[\alpha(r - r_1)]\exp[-\alpha^2(r - r_1)^2/2],$$

where $N = (\alpha/\pi^{1/2}2^n n!)^{1/2}$ is the normalization factor, $\alpha = \sqrt{2Mu/w_1^2\hbar^2}$ and H_n is a Hermite polynomial.

The matrix elements for dipole transitions which are important in the given energy region were obtained by numerical integration.

The excited level width Γ was determined as follows. A nucleon which has been excited through the absorption of a gamma quantum may return to the ground level immediately by emitting a gamma quantum of the same energy. This process is represented by the radiative width Γ_γ given by[6]

$$\Gamma_\gamma = 2\frac{L+1}{L}\frac{(2l+1)(2j+1)(2l'+1)}{[(2L+1)!!]^2}$$
$$\times\left[C(ll'L;\ 00)\,W\left(jlj'l';\ \frac{1}{2}\ L\right)\right]^2\hbar\omega\frac{e^2}{\hbar c}k^{2L}m_L^2,$$

where l, j pertain to the final state, l', j' pertain to the initial state, L is the emission multipole order (in our case L = 1), C is a Clebsch-Gordan coefficient, W is a Racah coefficient, and m_L is the radial part of the transition matrix element.

The nucleus can also return to the ground state by means of a cascade gamma transition, which can be calculated roughly.[7] The width for medium and heavy nuclei at $E_\gamma \sim 8 - 9$ Mev is found to be $0.2 - 1.0$ ev, which is considerably smaller than Γ_γ for a direct transition to the ground level and can thus be neglected.

However, it is also possible for the given nucleon to be scattered by another nucleon in the ground state; the width Γ_{sc} corresponds to this possibility. In order to determine Γ_{sc} approximately we use the picture of nucleon-nucleon scattering in the Fermi gas model; in order of magnitude it is found that[8]

$$\Gamma_{sc} \approx v_1\rho\sigma\hbar\{(E_1 - E_F)/E_F\}^2;\quad E_1 > E_F.$$

Here v_1 is the velocity of the excited nucleon, σ is the cross section for scattering by another nucleon, ρ is the density of nucleons which can efficiently scatter the excited nucleon. The last factor in the expression for Γ_{sc} takes the exclusion principle into account; all states below the Fermi energy E_F (in our case $E_F = 32$ Mev) are filled, with vacant states above this level.

The nucleon-nucleon scattering cross section σ was calculated using results obtained on the semiphenomenological theory with which Signell and Marshak[1] gave a good account of experimental findings. In the energy interval of interest here ($6 - 9$ Mev) the scattered phases in the cross section were calculated by extrapolating the results of this theory.

The total width of the excited level is thus $\Gamma = \Gamma_\gamma + \Gamma_{sc}$. The following results were obtained when the foregoing method was applied to resonance scattering of gamma rays on Ni^{58}, Cu^{63}, Pb^{208}, Bi^{209}, Sn^{118} and I^{127}.

In the case of the Ni^{58} nucleus the proton transition $1f_{7/2} \rightarrow 1g_{9/2}$ is important. A proton in the excited state $1g_{9/2}$ undergoes $1P_1$ scattering* on a neutron in the ground state $1f_{7/2}$. Experiments actually measure the integrated cross section

$$\overline{\sigma(\gamma,\gamma)} = \frac{1}{D}\int\sigma(\gamma,\gamma)\,dE_\gamma$$

averaged over the energy interval (integration is performed over the interval of interest, which in our case is of the order of 1 Mev). Taking this into account, at $E_\gamma = 7$ Mev we obtain $\overline{\sigma(\gamma,\gamma)} = 2.6$ mbn, which is in good agreement with experiment.

Cu^{63} differs from Ni^{58} by the addition of one proton in the $2p_{3/2}$ state and four neutrons, of which two are in the $2p_{3/2}$ state and two in the $1f_{5/2}$ state. An excited $1g_{9/2}$ proton can also be scattered by the latter neutrons, thus increasing Γ. The cross section corresponding to the proton transition $1f_{7/2} \rightarrow 1g_{9/2}$ is thus reduced, becoming 2.5 mbn. The experimental result is $\overline{\sigma(\gamma,\gamma)} \leq 1.8$ mbn.

The most efficient transition in Pb^{208} is $2f_{7/2} \rightarrow 2g_{9/2}$. The excited neutron is scattered by the $1h_{11/2}$ proton; the calculation gives $\overline{\sigma(\gamma,\gamma)} = 16$ mbn.

The calculation for Bi^{209} is practically the same as for Pb^{208} since only one proton in the $1h_{9/2}$ state is added, which is without appreciable effect. The cross section is the same as for Pb^{208} and the results for Pb^{208} and Bi^{209} are in good agreement with experiment. The efficient transition in Sn^{118} and I^{127} is $1f_{5/2} \rightarrow 1g_{7/2}$ (ΔE

*It should be noted that only P scattering is important. It can be shown by a direct calculation that S scattering, although possible, makes a negligible contribution since it corresponds to a very large value of Γ_{sc} (tens of kev).

~ 7 Mev). The width of the excited state is given by $\Gamma_\gamma \sim 325$ ev ($\Gamma_{sc} \sim 1$ ev). $\overline{\sigma(\gamma, \gamma)}$ for Sn^{118} is 12.5 mbn, which corresponds to the measured value. In I^{127} there is practically no change in the basic features of the $1f_{5/2} \rightarrow 1g_{7/2}$ transition. However, three protons are in the $1g_{7/2}$ state; there are fewer possibilities for a transition and the probability is diminished. The calculation of $\overline{\sigma(\gamma, \gamma)}$ gives 4.8 mbn, which can be compared with the experimental $\overline{\sigma(\gamma, \gamma)} < 3$ mbn. Thus, just as for the Cu^{63} nucleus, the calculation indicates that the cross section tends to decrease, although not so sharply as occurs experimentally.

The crude model that we are using must be regarded as only a first approximation, although it enables us to derive all of the basic features of gamma-ray scattering on nuclei in the subthreshold region. Relatively good agreement with experiment is obtained for both the cross section and its variation as we pass from a nucleus with closed shells to another which possesses extra nucleons.

Our theory has assumed a spherically symmetrical nucleus, although many nuclei are actually deformed. In most cases this deformation is small and its effect on gamma-ray scattering can be calculated by the method of Moszkowski,[9] who showed that for small deformations an energy level can be described by

$$E = E^{(0)} + E^{(1)}d + E^{(2)}d^2 + \ldots$$

where $d = \frac{2}{3}\epsilon$, and ϵ is the ratio of the difference between the semiaxes of the ellipsoid describing the nucleus to the radius of the equivalent sphere; $E^{(1)}_{nljm}$ is defined by

$$E^{(1)}_{nljm} = -2E^{(0)}_{nljm}\frac{j(j+1)-3m^2}{2j(2j+2)}f.$$

Here $E^{(0)}_{nljm}$ is the energy in the case of spherical symmetry and f is a coefficient which in our case of an oscillator type of potential equals one-half.

Confining ourselves to first order terms in the expansion of E with respect to powers of the deformation d and considering as a specific example the proton transition $1f_{5/2} \rightarrow 1g_{7/2}$ in I^{127} (assuming for simplicity that $\Delta m = 0$), we obtain $d = 0.02$ (the quadrupole moment of I^{127} is 46×10^{-26} cm^2 and $R = 1.2A^{1/3} \times 10^{-13}$ cm). Instead of a single resonance at $E_\gamma = 7$ Mev we obtain three less strong resonances (since the levels are now only doubly degenerate); one of these is 33.6 kev above, while the other two are 3.8 kev and 92.6 kev below, the previous resonance. The splitting is much larger than the resonance widths (~ 300 ev).

On the basis of the foregoing we note that the maximum of the subthreshold resonance need not necessarily coincide with the threshold of the (γ, n) and (γ, p) reactions but may be slightly shifted toward lower energies. This effect depends to a considerable degree on the specific distribution of the levels between which radiative transitions occur.

In conclusion the author wishes to thank B. T. Geĭlikman for his interest and valuable suggestions.

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THE PROBLEM OF THE OPTICAL CONSTANTS OF CONDUCTORS

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We discuss the problem of determining the complete set of optical constants of a conductor. It is shown that for an isotropic conductor this set consists not only of the index of refraction and the absorption coefficient, but also of two real quantities corresponding to a complex boundary impedance. The real part of the boundary impedance determines the surface losses in the conductor, while the imaginary part of the dielectric constant determines the volume losses. We have formulated dispersion relations which connect the real and the imaginary parts of the complex surface conductivity. We have considered fluctuations in the electromagnetic field in the conductor and have obtained the correlation functions for the field components of a metal filling a half-space.

1. In a preceding paper by the author[1] the theory of the optical constants of conductors was considered for the case of radiation obliquely incident on the surface of a bulk, conducting body. We evaluated then the anomalous skin-effect by introducing a boundary condition at the surface of the conductor corresponding to the presence of a surface current. The surface conductivity evaluated in that way leads to additional losses corresponding to a diffusive scattering of the conduction electrons at the metal surface. In the following we shall discuss the problem of the optical constants of a conductor and we shall consider fluctuations in the electromagnetic field and pay attention to some electrodynamical relations, taking into account the presence of a boundary impedance $z = \gamma^{-1}$ which connects the surface current density $\mathbf{i}$ with the tangential components of the electrical field:

$$\mathbf{i} = \gamma(\omega)\{\mathbf{E} - \mathbf{n}(\mathbf{nE})\}, \tag{1}$$

$\gamma(\omega)$ is the complex surface conductivity, $\mathbf{n}$ the outward normal to the surface of the conducting body. We have taken here the time dependence in the form $e^{i\omega t}$. We have for an anisotropic conductor, and also for a crystal of cubic symmetry, instead of (1)

$$i_\alpha = \gamma_{\alpha\beta} E_\beta, \tag{2}$$

where $\gamma_{\alpha\beta}$ is the second-order conductivity tensor.

2. If we do not take the anomalous skin effect into account, the optical properties of conductors are characterized by a complex dielectric constant

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \varepsilon(\omega) = (n - i\varkappa)^2 = \varepsilon_1 + i\varepsilon_2. \tag{3}$$

Here n is the index of refraction and κ the absorption coefficient. With the same degree of accuracy with which we may neglect the anomalous skin effect, these quantities fully determine the optical properties of a conductor. In the region of the anomalous skin effect, on the other hand, when the mean free path is larger than or comparable to the penetration depth of the field, the complex dielectric constant, and thus n and κ, do no longer determine all the optical constants of the conductor. The problem therefore arises of determining the complete set of optical constants in that region. There is in the literature essentially no elucidation of this problem.

The majority of the papers considering the anomalous skin effect in the optical region are devoted to a microtheory and because of the complexity of the equations of the microfield they are restricted to a study of the case of bulk metals.* The discussion is appreciably simplified in the case when the absolute magnitude of the complex dielectric constant is appreciably larger than unity. One can then use Leontovich's boundary conditions[2] and they lead in such a limiting case to the fact that the optical properties of a bulk metal are completely characterized by the complex surface impedance (see reference 3). Moreover, it turns out that one can introduce an effective complex dielectric constant and thus n_{eff} and κ_{eff}.[4]

*See the papers quoted in reference 1.

These quantities characterize the conductor, however, only in the region where ϵ_{eff} is large, and only for the case where the dimensions of the conductor are large compared to the skin-depth δ, i.e., for bulk conductors. As an example we point out that the electrical polarizability of a sphere of radius R which determines the scattering and absorption of light in the case where R is small compared to the wavelength of the radiation both in a cavity and also inside the particle is equal to

$$\alpha^{(e)} = \frac{3}{4\pi}\frac{\tilde{\varepsilon}-1}{\tilde{\varepsilon}+2}, \qquad \text{where } \tilde{\varepsilon} = \varepsilon - \frac{8\pi i\gamma(\omega)}{\omega R}.$$

One can use the complex dielectric constant in optics in the region of the anomalous skin effect. This is because even in the infrared region the condition $v/\omega \ll \delta$, where v is the velocity of the conduction electrons, is satisfied. If we are interested also in terms of the order $(v/\omega\delta)^2$ we must take into account the spatial dispersion of the dielectric constant and instead of Eq. (3) use an equation of the form[5]

$$\mathbf{D} = \varepsilon\mathbf{E} + a\nabla^2\mathbf{E} + b\,\text{grad div}\,\mathbf{E}. \tag{4}$$

Since the boundary impedance corresponds to an effect of the first order in $v/\omega\delta$ we need take the anomalous skin effect correction into account only when for some reason or other γ turns out to be excessively small. This can, in particular, occur in the case of specular reflection of the conduction electrons from the metal surface, a case which is of little practical interest. In metals, however, diffusive scattering takes apparently always place and γ is by no means abnormally small. Because of this one can with great accuracy use Eq. (3), i.e., neglect the spatial dispersion of the dielectric constant. If γ is not abnormally small the spatial dispersion can be appreciable only for large $v/\omega\delta$ when any expansion of (4) in powers of such a parameter becomes invalid.*

In essence the complete answer to the problem of the optical constants of a metal when the anomalous skin effect is taken into account is contained in what has been said. In addition to the complex dielectric constant there enters namely, into the complete set of optical constants also a complex boundary impedance, i.e., two real number γ_1 and γ_2 $(\gamma = \gamma_1 + i\gamma_2)$, which are, like n and κ, functions of the frequency of the electromagnetic field.

*The case when ε tends to zero is an exception. This case corresponds to the possibility of the propagation of plasma waves. The velocity of propagation of these waves is essentially determined by the spatial dispersion of the dielectric constant.

The optical properties of a conductor are then completely characterized by ϵ and γ, independent of whether or not ϵ is large. We note that the boundary impedance can in any case not be neglected (or rather, its real part cannot be neglected) when the imaginary part of ϵ is small, which occurs when $\delta \lesssim l$, where l is the conduction electron mean free path.

In the case of an anisotropic conductor we have instead of (3)

$$D_\alpha = \varepsilon_{\alpha\beta}E_\beta, \tag{5}$$

which gives us together with (2) the complete set of material equations which determine the optical properties of metallic crystals.

3. We shall consider some consequences of the Maxwell equations and the material equations given above which enable us to obtain useful relations for optics when the anomalous skin effect is taken into account. First of all we consider the boundary conditions at the interface of the conductor with a vacuum (the generalization to the case of an interface of a conductor with a dielectric or with another conductor is obvious). The presence of a surface current leads to the following boundary condition for the tangential components of the magnetic field (we shall take $\mu = 1$):

$$\text{Curl}\,\mathbf{H} \equiv \mathbf{n}\times[\mathbf{H}^{\text{med}} - \mathbf{H}^{\text{vac}}] = 4\pi\mathbf{i}/c. \tag{6}$$

The component of the magnetic field normal to the surface as well as the tangential components of the electrical field are continuous

$$H_n^{\text{vac}} = H_n^{\text{med}}, \quad E_t^{\text{vac}} = E_t^{\text{med}}. \tag{7}$$

The normal component of the electrical induction, on the other hand, is discontinuous because the surface current density $\mathbf{i}$ leads by virtue of the equation of continuity to the presence of a surface charge density. If we take the time dependence in the form $e^{i\omega t}$, we get for the surface charge density $\sigma = (i/\omega)\,\text{div}\,\mathbf{i}$. The last boundary condition is therefore of the form

$$D_n^{\text{vac}} - D_n^{\text{med}} = (4\pi i/\omega)\,\text{div}\,\mathbf{i}. \tag{8}$$

Because the vector $\mathbf{i}$ lies in the surface, there is in the first part of this relation no normal derivative, and only the tangential derivatives of the tangential components of the electrical field enter therefore according to (1) or (2) and these are continuous because the normal components of the magnetic field are also continuous.

Using the boundary conditions (6) — (8) we can determine the heat released inside the conductor. The energy current flowing through the surface of the conductor is namely equal to

$$-\oint dS\frac{c}{4\pi}(\mathbf{n}[\mathbf{E}^{\mathrm{vac}}\times\mathbf{H}^{\mathrm{vac}}]$$

$$=\int dV\frac{1}{4\pi}\left(\mathbf{E}\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\frac{\partial B}{\partial t}\right)+\oint dS(\mathbf{iE}), \qquad (9)$$

where the first integral is taken over the whole volume of the conducting body, and the second one over its surface. Taking the time average and using Eq. (1) we get for the average quantity of heat Q released in the conductor per unit time the following expression (we neglect the imaginary part of the magnetic susceptibility):

$$Q=-\frac{\omega\varepsilon_2}{4\pi}\int dV\,\overline{\mathbf{E}^2}+\gamma_1\oint dS\{\overline{\mathbf{E}^2}-\overline{(\mathbf{nE})^2}\}. \qquad (10)$$

We see that if we neglect the imaginary part of the magnetic susceptibility, the energy absorption is determined by the real part of γ (surface losses) and the imaginary part of ϵ (volume losses). It follows in particular from this that while $\epsilon_2<0$, γ_1 must be positive.

In the case of an anisotropic conductor we have instead of (10)

$$Q=-\frac{i\omega}{8\pi}\int dV(\varepsilon^*_{ik}-\varepsilon_{ki})\overline{E_iE^*_k}$$

$$+\frac{1}{2}\oint dS(\gamma^*_{ik}+\gamma_{ki})\overline{E_iE^*_k}. \qquad (11)$$

4. We considered above the surface current (1) for the components of the electromagnetic field with a well determined frequency ω taking into account that $\gamma(\omega)$ is some function of the frequency. The possibility of such a dispersion of γ follows, generally speaking, already from the fact that, for instance, in the isotropic case the surface current density at a given moment is determined in the usual way by the values of the field at the preceding moment (see, for instance, reference 3). One can thus show that $\gamma_1(\omega)$ is an even and $\gamma_2(\omega)$ an odd function of the frequency ω.

One can easily obtain the dispersion relations which connect the real and imaginary parts of the boundary impedance. To do this one can, for instance, follow the derivation given in reference 3 for obtaining the dispersion relations for the complex dielectric constant. The result is

$$\gamma_1(\omega)=\frac{2}{\pi}\int_0^\infty\frac{x\gamma_2(x)}{x^2-\omega^2}dx,\quad \gamma_2(\omega)=\frac{2\omega}{\pi}\int_0^\infty\frac{\gamma_1(x)}{x^2-\omega^2}dx. \qquad (12)$$

In the neighborhood of $x^2=\omega^2$ one must take the principal values of the integral in (12) as is usual in dispersion relations. To obtain the dispersion relations it is necessary to take into account that $\gamma(\omega)$ has no singularity as $\omega\to 0$. This is in accordance with the fact that γ is finite in the static limit. Apart from this $\gamma(\omega)$ must decrease when the frequency increases. One can verify that this condition is also satisfied by using the free electron model which is, generally speaking, the better applicable, the higher the frequency of the variable field.

Indeed, the surface current density produced by the collisions of the electrons with the surface of the conductor and their subsequent diffusive scattering is as far as order of magnitude is concerned equal to $\mathbf{i}=\mathrm{enL}\Delta\mathbf{v}w$, where e is the electron charge, n the number of electrons per unit volume, L the depth of the surface layer in which the electrons, which undergo noticeable collisions with the surface, lie. It is evident that under the conditions when the frequency of the variable field is large compared to the collision frequency of the electrons, L will be equal as to order of magnitude to the distance traversed by an electron during one period of the variable field, i.e., $L\sim v/\omega$ (where v is the root mean square velocity of the electrons). Furthermore, w, the probability for the diffusive scattering of the electrons, is never larger than unity. Finally, the change in the electron velocity under the influence of the variable field is as to order of magnitude (not taking phase amplification into account) equal to $\Delta\mathbf{v}\sim(e/m\omega)\mathbf{E}$ according to the equations of motion of a free electron. It is thus clear that $\gamma\lesssim e^2nv/m\omega^2$. A more rigorous evaluation, also based upon the free particle model but using the transport equation, gives $\gamma=3e^2nv_0/16m\omega^2$, where v_0 is the velocity on the Fermi surface and m the effective mass.

The generalization of Eq. (12) for the anisotropic case is completely obvious.

5. We shall now consider the problem of the fluctuations in the electromagnetic field when there is a complex surface resistivity. It is usual when considering fluctuations in the electromagnetic field to introduce a "strange" fluctuating induction corresponding to spontaneous electrical and magnetic moments produced in the body as a result of the fluctuating charge oscillations (see reference 6 and Chap. XIII of reference 3). In accordance with this, the material equations take the form

$$D_\alpha(\omega)=\varepsilon_{\alpha\beta}(\omega)E_\beta(\omega)+K_\alpha(\omega),$$

$$B_\alpha(\omega)=\mu_{\alpha\beta}(\omega)H_\beta(\omega)+L_\alpha(\omega), \qquad (13)$$

and from the Maxwell equations we get

$$K_\alpha(\omega)=-\varepsilon_{\alpha\beta}E_\beta(\omega)-\frac{ic}{\omega}\,\mathrm{curl}_\alpha\mathbf{H}(\omega),$$

$$L_\alpha(\omega)=-\mu_{\alpha\beta}H_\beta(\omega)+\frac{ic}{\omega}\,\mathrm{curl}_\alpha\mathbf{E}(\omega). \qquad (14)$$

The correlation between the strange inductions is then determined by the equations (our notation

differs from the one in reference 3 by the opposite sign of the frequency ω)

$$(K_\alpha(\mathbf{r}_1)K_\beta(\mathbf{r}_2))_\omega = i\hbar(\varepsilon_{\alpha\beta} - \varepsilon^*_{\beta\alpha})\delta(\mathbf{r}_2 - \mathbf{r}_1) \times \coth(\hbar\omega/2\varkappa T), \text{ and so on} \quad (15)$$

Here $\varkappa$ is the Boltzmann constant. The formulae which we have written down solve in principle the problem of determining the electromagnetic fluctuations in any body, if there are no surface currents present.

It is also in our case natural to supplement the material Eq. (2) with a fluctuating surface current density $\mathbf{g}$:

$$i_\alpha(\omega) = \gamma_{\alpha\beta}(\omega)E_\beta(\omega) + g_\alpha(\omega). \quad (16)$$

Substituting this material equation into the boundary condition (6) we obtain the following relation in addition to Eqs. (14):

$$g_\alpha(\omega) = -\gamma_{\alpha\beta}(\omega)E_\beta(\omega) + (c/4\pi)\operatorname{curl}_\alpha \mathbf{H}(\omega). \quad (17)$$

One can say that the fluctuating surface current density $\mathbf{g}$ is caused by the fluctuations in the scattering of the electrons at the metal surface.

Equation (17) is analogous to the first equation of (14). In order that we can use the known results of the theory of electromagnetic fluctuations it is convenient to construct a symmetric scheme and to introduce a surface equation similar to the second equation of (14). The possibility of formulating such an equation can physically be connected, for instance, with the fact that the probability for an electron spin flip at the surface of the conductor is different from zero (see, for instance, reference 7). We can thus write

$$\operatorname{Curl}\mathbf{E} \equiv \mathbf{n}\times[\mathbf{E}^{\text{med}} - \mathbf{E}^{\text{vac}}] = -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial t}, \quad (18)$$

where $\mathbf{b}/4\pi$ is the surface magnetization density. Introducing the strange fluctuating magnetic induction density l, we have

$$b_\alpha = \nu_{\alpha\beta}(\omega)\tfrac{1}{2}(H^{\text{vac}}_\beta + H^{\text{med}}_\beta) + l_\alpha, \quad (19)$$

$$l_\alpha = -\tfrac{1}{2}\nu_{\alpha\beta}(H^{\text{vac}}_\beta + H^{\text{med}}_\beta) + \frac{ic}{\omega}\mathbf{n}\times[\mathbf{E}^{\text{med}} - \mathbf{E}^{\text{vac}}]_\alpha. \quad (20)$$

Since according to (18) the tangential components of the electrical field are discontinuous we must replace in Eqs. (16) and (17) $\mathbf{E}$ by, for instance, $(\mathbf{E}^{\text{vac}} + \mathbf{E}^{\text{med}})/2$. Moreover, it is convenient to introduce instead of the fluctuating surface current density $\mathbf{g}$ a fluctuating surface electrical induction $\mathbf{k}$. We get then

$$k_\alpha(\omega) = -\tfrac{1}{2}\eta_{\alpha\beta}(\omega)(E^{\text{vac}}_\beta + E^{\text{med}}_\beta) - \frac{ic}{\omega}\mathbf{n}\times[\mathbf{H}^{\text{med}} - \mathbf{H}^{\text{vac}}]_\alpha \quad (21)$$

where

$$k_\alpha(\omega) = (4\pi/i\omega)g_\alpha(\omega), \qquad \eta_{\alpha,\beta} = (4\pi/i\omega)\gamma_{\alpha\beta}. \quad (22)$$

Relations (20) and (21) are completely analogous to the volume relations (14). The form in which they are written makes it therefore immediately possible to write down the following formulae for the correlations between the incidental surface inductions:

$$(k_\alpha(\mathbf{r}_1)k_\beta(\mathbf{r}_2))_\omega = i\hbar(\eta_{\alpha\beta} - \eta^*_{\beta\alpha})\delta([\mathbf{r}_2 - \mathbf{r}_1]\times\mathbf{n})\coth(\hbar\omega/2\varkappa T),$$

$$(l_\alpha(\mathbf{r}_1)l_\beta(\mathbf{r}_2))_\omega = i\hbar(\nu_{\alpha\beta} - \nu^*_{\beta\alpha})\delta([\mathbf{r}_2 - \mathbf{r}_1]\times\mathbf{n})\coth(\hbar\omega/2\varkappa T),$$

$$(k_\alpha l_\beta)_\omega = 0. \quad (23)$$

In Eqs. (23) there occur surface δ-functions and not volume δ-functions, in contradistinction to Eq. (15). This is, of course, connected with the fact that in the expressions for the change in energy k and l enter into a surface integral which is taken over the boundary surface of the body. Taking (22) into account we have the following expression for the correlation between the incidental surface currents:

$$(g_\alpha(\mathbf{r}_1)g_\beta(\mathbf{r}_2)) = \frac{\hbar\omega}{4\pi}(\gamma_{\alpha\beta} + \gamma^*_{\beta\alpha})\delta([\mathbf{r}_2 - \mathbf{r}_1]\times\mathbf{n})\coth(\hbar\omega/2\varkappa T). \quad (24)$$

From Eqs. (23), (24) and the Onsager relations in the case of bodies which do not possess a "magnetic structure"[3] and which are not in an external magnetic field the following equations follow

$$\nu_{\alpha\beta} = \nu_{\beta\alpha}, \qquad \eta_{\alpha\beta} = \eta_{\beta\alpha}, \qquad \gamma_{\alpha\beta} = \gamma_{\beta\alpha}. \quad (25)$$

If, however, the system is in a constant magnetic field $\mathbf{H}$, we have

$$\gamma_{\alpha\beta}(\mathbf{H}) = \gamma_{\beta\alpha}(-\mathbf{H}). \quad (26)$$

6. Equations (21) — (23) together with Eqs. (14) and (15) make it in principle possible to determine the fluctuations in the electromagnetic field in a conductor in the region where one can apply the material equations used by us. We shall in the following consider as an example the fluctuations in the electromagnetic field in an isotropic metal in the case where we can neglect the imaginary part of the dielectric constant and the magnetic susceptibility, but where it is necessary to take the real part of γ into account. Such a case corresponds just to the region of the anomalous skin-effect. One can then drop the strange fluctuating volume inductions, since it is evident that the

fluctuations in the field will be caused only by the incidental surface current density $\mathbf{g}$.

Assuming that the metal fills the half-space $z < 0$, we can write down the system of basic equations necessary to solve our problem in the following form

$$\begin{aligned} \operatorname{curl} \mathbf{E}(\omega) &= i(\omega/c)\mathbf{H}(\omega), \\ \operatorname{curl} \mathbf{H}(\omega) &= -i(\omega/c)\mathbf{E}(\omega), \qquad z > 0, \\ \operatorname{curl} \mathbf{E}(\omega) &= i(\omega/c)\mathbf{H}(\omega), \\ \operatorname{curl} \mathbf{H}(\omega) &= -i(\omega/c)\varepsilon\mathbf{E}(\omega), \qquad z < 0, \end{aligned} \tag{27}$$

$$(c/4\pi)\,\mathbf{n}\times[\mathbf{H}^{\mathrm{med}} - \mathbf{H}^{\mathrm{vac}}] = \gamma(\mathbf{E} - \mathbf{n}(\mathbf{En})) + \mathbf{g}, \qquad z = 0. \tag{28}$$

Solving the Maxwell equations (27) with the boundary conditions (28), using, for instance, Fourier transforms, we can express the electromagnetic field in the form of a functional of the fluctuating surface current density which is afterwards averaged by means of (24). We give the final relations which determine the spatial correlations of the components of the electrical field strength:

$$\overline{E_\alpha(\mathbf{r}_1,\omega)E^*_\beta(\mathbf{r}_2,\omega')} = \delta(\omega+\omega')(E^{(1)}_\alpha E^{(2)}_\beta)_\omega;$$

$$\begin{aligned} (E^{(1)}_\alpha E^{(2)}_\beta)_\omega &= \frac{2\hbar\omega^3}{c^4}\coth\frac{\hbar\omega}{2\varkappa T} \\ &\times \operatorname{Re}\gamma\int_0^\infty dq\,\frac{\exp\{i(\omega/c)[z_1\sqrt{\varepsilon(\omega)-q} - z_2\sqrt{\varepsilon(-\omega)-q}]\}}{|\sqrt{1-q}+\sqrt{\varepsilon-q}+(4\pi\gamma/c)|^2} \\ &\times\left\{\left[\frac{1}{2}(1+|1-qA|^2)(\delta_{\alpha\beta}-\delta_{\alpha z}\delta_{\beta z}) + \frac{|1-qA|^2}{|\varepsilon-q|}q\delta_{\alpha z}\delta_{\beta z}\right]\right. \\ &\times J_0\left(\frac{\omega}{c}R\sqrt{q}\right) - i\sqrt{q}|1-qA|^2 \\ &\times\left[\sqrt{\varepsilon(-\omega)-q}\,\frac{R_\alpha}{R}\delta_{\beta z}(1-\delta_{\alpha z})\right. \\ &+\sqrt{\varepsilon(\omega)-q}\,\frac{R_\beta}{R}\delta_{\alpha z}\,(1-\delta_{\beta z})\Big]J_1\left(\frac{\omega}{c}R\sqrt{q}\right) \\ &+(1-|1-qA|^2) \\ &\times\left[\frac{R_\alpha R_\beta}{R^2}(1-\delta_{\alpha z})(1-\delta_{\beta z})\right. \\ &\left.\left.-\frac{1}{2}(\delta_{\alpha\beta}-\delta_{\alpha z}\delta_{\beta z})\right]J_2\left(\frac{\omega}{c}R\sqrt{q}\right)\right\}, \qquad z_1, z_2 < 0; \end{aligned}$$

$$\begin{aligned} (E^{(1)}_\alpha E^{(2)}_\beta)_\omega &= \frac{2\hbar\omega^3}{c^4}\coth\frac{\hbar\omega}{2\varkappa T}\operatorname{Re}\gamma \\ &\times\int_0^\infty dq\,\frac{\exp\{-i(\omega/c)[z_1\sqrt{1-q} - z_2(\sqrt{1-q})^*]\}}{|\sqrt{1-q}+\sqrt{\varepsilon-q}+(4\pi\gamma/c)|^2} \\ &\times\left\{\left[\frac{1}{2}(1+|1-qA|^2)(\delta_{\alpha\beta}-\delta_{\alpha z}\delta_{\beta z}) + \frac{|1-qA|^2}{|1-q|}q\delta_{\alpha z}\delta_{\beta z}\right]\right. \\ &\times J_0\left(\frac{\omega}{c}R\sqrt{q}\right) - i\sqrt{q}|1-qA|^2 \\ &\times\left[(\sqrt{1-q})^*\,\frac{R_\alpha}{R}\delta_{\beta z}(1-\delta_{\alpha z})\right. \\ &+\sqrt{1-q}\,\frac{R_\beta}{R}\delta_{\alpha z}(1-\delta_{\beta z})\Big]J_1\left(\frac{\omega}{c}R\sqrt{q}\right) \\ &+(1-|1-qA|^2)\left[\frac{R_\alpha R_\beta}{R^2}(1-\delta_{\alpha z})(1-\delta_{\beta z})\right. \\ &\left.\left.-\frac{1}{2}(\delta_{\alpha\beta}-\delta_{\alpha z}\delta_{\beta z})\right]J_2\left(\frac{\omega}{c}R\sqrt{q}\right)\right\}, \qquad z_1 > 0, \quad z_2 < 0. \end{aligned}$$

$$\begin{aligned} (E^{(1)}_\alpha E^{(2)}_\beta)_\omega &= \frac{2\hbar\omega^3}{c^4}\coth\frac{\hbar\omega}{2\varkappa T}\operatorname{Re}\gamma\int_0^\infty dq \\ &\times\frac{\exp\{-i(\omega/c)[z_1\sqrt{1-q}+z_2\sqrt{\varepsilon(-\omega)-q}]\}}{|\sqrt{1-q}+\sqrt{\varepsilon-q}+(4\pi\gamma/c)|^2} \\ &\times\left\{\left[\frac{1}{2}(1+|1-qA|^2)(\delta_{\alpha\beta}-\delta_{\alpha z}\delta_{\beta z})\right.\right. \\ &\left.+\frac{|1-qA|^2}{\sqrt{1-q}\sqrt{\varepsilon(-\omega)-q}}q\delta_{\alpha z}\delta_{\beta z}\right]J_0\left(\frac{\omega}{c}R\sqrt{q}\right) \\ &-i\sqrt{q}|1-qA|^2\left[\sqrt{\varepsilon(-\omega)-q}\,\frac{R_\alpha}{R}\delta_{\beta z}(1-\delta_{\alpha z})\right. \\ &+\sqrt{1-q}\,\frac{R_\beta}{R}\delta_{\alpha z}(1-\delta_{\beta z})\Big]J_1\left(\frac{\omega}{c}R\sqrt{q}\right) \\ &+(1-|1-qA|^2)\left[\frac{R_\alpha R_\beta}{R^2}(1-\delta_{\alpha z})(1-\delta_{\beta z})\right. \\ &\left.\left.-\frac{1}{2}(\delta_{\alpha\beta}-\delta_{\alpha z}\delta_{\beta z})\right]J_2\left(\frac{\omega}{c}R\sqrt{q}\right)\right\}, \qquad z_1 > 0, \quad z_2 < 0. \end{aligned} \tag{29}$$

Here $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$, J_n is a Bessel function, and A is defined by the formula

$$A = \left\{\frac{1}{\sqrt{1-q}}+\frac{1}{\sqrt{\varepsilon-q}}\right\}\left\{\frac{1}{\sqrt{1-q}}+\frac{\varepsilon}{\sqrt{\varepsilon-q}}+\frac{4\pi}{c}\gamma\right\}^{-1}.$$

There is no summation over repeated indices in Eqs. (29). One can find the correlation formulae for the magnetic field also by completely analogous means.

Equations (29) enable us, in particular, to obtain the following expression for the density of the energy of the electromagnetic field outside the metal:

$$\begin{aligned} w_\omega &= \frac{1}{4\pi}(\mathbf{E}^2(\mathbf{r})+\mathbf{H}^2(\mathbf{r}))_\omega = \frac{\hbar\omega^3}{\pi c^4}\coth\frac{\hbar\omega}{2\varkappa T}\operatorname{Re}\gamma\left\{\int_0^1 dq\right. \\ &+\int_1^\infty dq\cdot q\exp\left(-2\frac{\omega}{c}z\sqrt{q-1}\right) \\ &\times\left[\left|\sqrt{1-q}+\sqrt{\varepsilon-q}+\frac{4\pi\gamma}{c}\right|^{-2}\right. \\ &\left.\left.+\left|1+\varepsilon\sqrt{\frac{1-q}{\varepsilon-q}}+\frac{4\pi\gamma}{c}\sqrt{1-q}\right|^{-2}\right]\right\}. \end{aligned} \tag{30}$$

The integral over q from zero to unity gives the radiation energy density which does not depend on the distance and corresponds to the energy of the wave field. The second integral over q from unity to infinity in Eq. (30) corresponds to the energy of the quasi-stationary field which de-

creases away from the surface of the conductor. At large distances this diminution is determined by the relation

$$w_{\omega\,\mathrm{quasi}} \approx \frac{\hbar\omega^3}{2\pi c^4} \coth\frac{\hbar\omega}{2\varkappa T}\,\mathrm{Re}\,\gamma \times\left\{1+\frac{1}{|\sqrt{\varepsilon-1}+4\pi\gamma/c|^2}\right\}(c/\omega z)^2. \qquad (31)$$

In the case where the absolute magnitude of the dielectric constant is large compared to unity the expression for the energy density of the quasi-stationary field can essentially be obtained directly from Eq. (1.18) of Rytov's book[6] if one substitutes in that equation for the dielectric constant $\epsilon_{\mathrm{eff}} = (\sqrt{\epsilon}+4\pi\gamma/c)^2$. This possibility is caused by the fact that in the region of large values of ϵ the presence of a surface current can be taken into account by introducing just such an effective dielectric constant.[1]

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RESONANCE ABSORPTION OF ULTRASOUND ON NUCLEI

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An analysis is given for resonance absorption of ultrasound on paramagnetic nuclei in a simple cubic lattice under the assumption that the spin-lattice interaction is due to nuclear quadrupole forces. Absorption factors for the spin transitions characterized by $\Delta m = 1$ and $\Delta m = 2$ are obtained for arbitrary directions of propagation and polarization of the acoustic waves. A comparison between theory and experiment is made for In^{115} in InSb.

1. INTRODUCTION

WHEN acoustic waves of frequency ν are propagated in a paramagnet located in a magnetic field H_0, resonance absorption of the sound can occur if the condition $h\nu = g\beta H_0$ is satisfied, where g is the splitting factor and β is the Bohr magneton.

An analysis of this problem was first given by Al'tshuler[1] who showed that the acoustic absorption is similar to first-order spin-lattice relaxation. Al'tshuler gives the following general formula for resonance absorption of ultrasound:

$$\sigma = \frac{4\pi^2}{h^2}\frac{NhvV}{kTv\nu_{1/2}n_\nu}|\langle m, n_\nu - 1 | \hat{\mathcal{H}}' | m', n_\nu\rangle|^2, \qquad (1)$$

where N is the number of magnetic particles per unit volume, v is the acoustic velocity, V and T are the volume and temperature of the paramagnet, $\nu_{1/2}$ is the half-width of the absorption line, and $\langle m, n_\nu - 1 | \mathcal{H}' | m', n_\nu\rangle$ is the matrix element for the spin-lattice interaction for the magnetic sub-levels m and m' and the lattice states $n_\nu - 1$ and n_ν. The absorption factors have been computed by Al'tshuler for a number of materials on the basis of the relaxation mechanisms proposed by Waller and Kronig and Van Vleck.

A number of experiments[2,3] have shown that nuclear spin transitions can be induced by ultrasonic waves. Menes and Bolef[4] have also observed resonance absorption of ultrasound on In^{115} nuclei in InSb.

In the present paper we consider the nuclear quadrupole relaxation mechanism and consider resonance absorption of ultrasound on nuclei.

2. SPIN-LATTICE INTERACTION OPERATOR

The energy of the electric quadrupole interaction of the nucleus with the electric field of the crystal lattice is given by[5]

$$\hat{\mathcal{H}}' = \sum_{\alpha=-2}^{2}(-1)^\alpha \hat{Q}_\alpha \gamma \nabla E^{-\alpha}, \qquad (2)$$

where

$$\hat{Q}_0 = A[3\hat{I}_z^2 - I(I+1)],$$

$$\nabla E^0 = \frac{1}{2}\sum_j e_j r_j^{-5}(3z_j^2 - r_j^2) = \sum_j \nabla E_j^0,$$

$$\hat{Q}_{\pm 1} = \pm\frac{\sqrt{6}}{2}A[\hat{I}_\pm \hat{I}_z + \hat{I}_z\hat{I}_\pm],$$

$$\nabla E^{\pm 1} = \pm\frac{\sqrt{6}}{2}\sum_j e_j r_j^{-5} z_j (x_j \pm iy_j) = \sum_j \nabla E_j^{\pm 1},$$

$$\hat{Q}_{\pm 2} = \frac{\sqrt{6}}{2}A\hat{I}_\pm^2, \quad \nabla E^{\pm 2} = \frac{\sqrt{6}}{4}\sum_j e_j r_j^{-5}(x_j \pm iy_j)^2 = \sum_j \nabla E_j^{\pm 2},$$

$$A = eQ/2I(2I-1),$$

$$eQ = \langle I, I \left| \sum_j e_j r_j^2 \{3\hat{I}_z^2 - I(I+1)\} \right| I, I\rangle,$$

$$\hat{I}_\pm = \hat{I}_x \pm i\hat{I}_y. \qquad (3)$$

$\hat{I}$ and $\hat{Q}$ are the operators for the spin and quadrupole moment of the nucleus, e_i and r_i are the charge of an elementary volume in the nucleus and its distance from the center of the nucleus, e_j and r_j are the charge of the neighboring ion and the distance between it and the nucleus, γ is a quantity which is introduced to take account of the following three factors: the increase in the interaction due to the large quadrupole moments induced in the electron shells of the atom by the quadrupole moment of the nucleus, the reduction in the interaction because of the polarization of the atom by the crystalline field, and the increase in the interaction because of the covalent bonding. It is proposed to determine the quantity γ experimentally.

Equation (2) does not take account of lattice vibrations. In order to take these vibrations into account r_j in Eq. (2) must be written in the form $R_j + u_{kj}$ where R_j is the equilibrium position of

the ion and $\mathbf{u}_{kj}$ is the relative displacement between the nucleus in question and the corresponding ion of the lattice. For simplicity we consider a cubic lattice and the six neighboring ions. Their equilibrium positions, in a coordinate system with origin at the nucleus and with axes parallel to the C_4 axis of the crystal, are at the points $\mathbf{R}_{1,2}(\pm a, 0, 0)$, $\mathbf{R}_{3,4}(0, \pm a, 0)$, $\mathbf{R}_{5,6}(0, 0, \pm a)$, where a is the lattice constant. The displacement of the ion from its equilibrium position is usually written in the form

$$u_k = \sum_{\varphi} \sum_{\delta=x,\,y,\,z} [C_{\varphi\delta} \cos(\mathbf{k}\varphi) + C'_{\varphi\delta} \sin(\mathbf{k}\varphi)]\, \mathbf{u}^0_\delta,$$

where φ is a vector normal to the wave, whose magnitude is given by $|\varphi| = 2\pi a\nu/v$; $\mathbf{k}$ is the position of the ion (components expressed in units of a); $\mathbf{u}^0_\delta$ is the normalized polarization vector.

The relative displacement of two neighboring interstices of the lattice will be:

$$\mathbf{u}_{k1} = \sum_{\varphi}\sum_{\delta} \varphi_x q_\delta(\varphi)\, \mathbf{u}^0_\delta, \qquad \mathbf{u}_{k3} = \sum_{\varphi}\sum_{\delta} \varphi_y q_\delta(\varphi)\, \mathbf{u}^0_\delta \text{ etc.}, \tag{4}$$

where $q_\delta(\varphi)$ characterizes propagation with wave number φ and polarization δ. The index k, which corresponds to the initial nucleus, will be dropped in the following. It should be recalled that $\mathbf{u}_1 = -\mathbf{u}_2$, $\mathbf{u}_3 = -\mathbf{u}_4$, $\mathbf{u}_5 = -\mathbf{u}_6$. As is well known[6]

$$\langle n_\nu - 1 \,|\, q_\delta(\varphi) \,|\, n_\nu\rangle = [n_\nu h / 4\pi M\nu]^{1/2}, \tag{5}$$

where M is the mass of the crystal.

Writing $\mathbf{r}_j = \mathbf{R}_j + \mathbf{u}_j$ we expand ΔE^α about the equilibrium distance between the ions $|\mathbf{R}_j| = a$. Limiting ourselves to first-order terms in $\mathbf{u}_j$, since we are interested only in the first-order processes:

$$\nabla E^0(\mathbf{u}) = \sum_j \nabla E^0_j(0) + \sum_j \sum_\delta [\partial(\nabla E^0_j)/\partial u_{j\delta}]_{r_j=R_j} u_{j\delta}, \tag{6}$$

$$[\partial(\nabla E^0_j)/\partial u_{jx}]_{r_j=R_j} = X_j[1 - Z_j^2 R_j^{-2}], \ \ldots\ . \tag{7}$$

Substituting (7) in (6), when $e_j = e$, we have

$$\nabla E^0(\mathbf{u}) = \nabla E^0(0) + 3ea^{-4}[u_{1x} + u_{3y} - 2u_{5z}],$$

and, similarly,

$$\nabla E^{\pm1}(\mathbf{u}) = \nabla E^{\pm1}(0) \pm (\sqrt{6}/2)\, ea^{-4}[u_{5x} \pm iu_{5y} + u_{1z} \pm iu_{3z}],$$

$$\nabla E^{\pm2}(\mathbf{u}) = \nabla E^{\pm2}(0) + (\sqrt{6}/4)\, ea^{-4}[3(u_{3y} - u_{1x}) \pm 2i(u_{3x} + u_{1y})]. \tag{8}$$

In what follows, in Eq. (8) we will drop the term $\nabla E^\alpha(0)$ since it causes only an unimportant shift in the levels. Using (2), (3), and (8) we write the interaction operator in the form

$$\hat{\mathcal{H}}'_0 = B[3\hat{I}_z^2 - I(I+1)][u_{1x} + u_{3y} - 2u_{5z}],$$
$$\hat{\mathcal{H}}'_{\pm1} = B[\hat{I}_\pm\hat{I}_z + \hat{I}_z\hat{I}_\pm][u_{5x} \pm iu_{5y} + u_{1z} \pm iu_{3z}],$$
$$\hat{\mathcal{H}}'_{\pm2} = {}^1/_2 B\hat{I}_\pm^2[3(u_{3y} - u_{1x}) \pm 2i(u_{1y} + u_{3x})], \tag{9}$$

where

$$B = 3e^2Q\gamma / 2I(2I-1)a^4.$$

$\hat{\mathcal{H}}'_0$ does not cause transitions in the spin system and must be discarded.

3. MATRIX ELEMENTS FOR THE SPIN-LATTICE INTERACTION AND THE ABSORPTION COEFFICIENT

To find the absorption factors we consider several cases; these correspond to various orientations between the direction of propagation of the acoustic waves and the magnetic field and the various possible polarizations of the acoustic oscillations.

Perpendicular Propagation of the Acoustic Waves

The magnetic field is along the z axis and the ultrasound propagates along the x axis. In this case only $\mathbf{u}_1 = -\mathbf{u}_2$ is different from 0. Equation (9) yields

$$\hat{\mathcal{H}}'_{\pm1} = B[\hat{I}_\pm\hat{I}_z + \hat{I}_z\hat{I}_\pm]u_{1z}, \quad \hat{\mathcal{H}}'_{\pm2} = {}^1/_2 B\hat{I}_\pm^2[-3u_{1x} \pm 2iu_{1y}].$$

a) For transitions of the spin system with $\Delta m = 1$ different from 0 the matrix elements correspond to oscillations polarized along the z axis:

$$\langle m, n_\nu - 1 \,|\, \hat{\mathcal{H}}'_{\pm1} \,|\, m \mp 1, n_\nu\rangle$$
$$= B\langle m \,|\, \hat{I}_\pm\hat{I}_z + \hat{I}_z\hat{I}_\pm \,|\, m \mp 1\rangle \langle n_\nu - 1 \,|\, u_{1z} \,|\, n_\nu\rangle.$$

From Eq. (4), for a given frequency and polarization it follows that $u_{1Z} = \varphi_X q_Z(\varphi)$. Then

$$\langle n_\nu - 1 \,|\, u_{1z} \,|\, n_\nu\rangle = (2\pi a\nu/v)[n_\nu h / 4\pi^2 M\nu]^{1/2},$$

and the matrix element

$$\langle m \,|\, \hat{I}_\pm\hat{I}_z + \hat{I}_z\hat{I}_\pm \,|\, m \mp 1\rangle = \xi$$
$$\equiv (2m-1)\sqrt{(I+m)(I-m+1)}.$$

Thus,

$$\langle m, n_\nu - 1 \,|\, \hat{\mathcal{H}}'_{\pm1} \,|\, m \mp 1, n_\nu\rangle = (Ba/v)\,\xi\,(n_\nu h\nu / M)^{1/2}. \tag{10}$$

Substituting (10) in (1) we find the absorption factor

$$\sigma^{\perp}_{1z} = PD\xi^2\nu^2 / T,$$

where

$$P = 9\pi^2 e^4 N_0 / k = 2.05 \cdot 10^4,$$
$$D = Q^2\gamma^2[I^2(2I-1)^2 a^6 v^3 \nu_{1/2}\mu]^{-1}, \quad d = \frac{M}{V},$$

$N_0 = Nd/\mu$ is Avogadro's number and μ is the molecular weight.

TABLE I

	I	$Q\cdot10^{26}$ cm^2	$a\cdot10^{8}$ cm	$v\cdot10^{-5}$ cm/sec	$10^{-4}\gamma^2$	ξ^2	η^2	$D\cdot10^{88}$
Br^{79} in LiBr	3/2	28	2.745	3.9	3	12	12	1.2
Br^{79} in KBr	3/2	28	3.29	3.39	3.4	12	12	0.51
Br^{79} in AgBr	3/2	28	2.88	3	20	12	12	6.1
I^{127} in KI	5/2	—46	3.52	2.9	25	80	40	0.69
In^{115} in InSb	9/2	117	2.23			576	144	

b) For transitions of the spin system with $\Delta m = 2$ different from 0 we have the following matrix elements: for x-polarization (cf. also reference 7)

$$\langle m, n_\nu - 1 \,|\, \hat{\mathcal{H}}'_{\pm 2} \,|\, m \mp 2, n_\nu \rangle = -(3Ba/2v)(n_\nu h\nu / M)^{1/2},$$

for y-polarization

$$\langle m, n_\nu - 1 \,|\, \hat{\mathcal{H}}'_{\pm 2} \,|\, m \mp 2, n_\nu \rangle = \pm (iBa/v)[n_\nu h\nu / M]^{1/2}\eta,$$

where $\eta^2 = (I+m)(I+m-1)(I-m+1)(I-m+2)$. In this case the absorption factors are

$$\sigma^{\perp}_{2x} = {}^9/_4 PD\eta^2\nu^2/T, \qquad \sigma^{\perp}_{2y} = PD\eta^2\nu^2/T.$$

Parallel Propagation of the Acoustic Waves

The magnetic field and the direction of propagation of the acoustic waves are along the z axis. Only u_5 is different from 0. Equation (9) now yields

$$\hat{\mathcal{H}}'_{\pm 1} = B[\hat{I}_\pm \hat{I}_z + \hat{I}_z \hat{I}_\pm][u_{5x} \pm iu_{5y}], \qquad \hat{\mathcal{H}}'_{\pm 2} \equiv 0.$$

Thus only transitions with $\Delta m = 1$ are possible. For x-polarization

$$\langle m, n_\nu - 1 \,|\, \hat{\mathcal{H}}'_{\pm 1} \,|\, m \mp 1, n_\nu \rangle = (Ba/v)(n_\nu h\nu / M)^{1/2}\xi,$$

for y-polarization

$$\langle m, n_\nu - 1 \,|\, \hat{\mathcal{H}}'_{\pm 1} \,|\, m \mp 1, n_\nu \rangle = \pm (iBa/v)(n_\nu h\nu / M)^{1/2}\xi.$$

The corresponding absorption factors are

$$\sigma^{\parallel}_{1x} = PD\xi^2\nu^2/T, \qquad \sigma^{\parallel}_{1y} = PD\xi^2\nu^2/T.$$

From these general formulas it is possible to find the numerical values of σ for a number of materials whose pertinent parameters are known. These parameters and the quantities ξ^2, η^2 and D are shown in Table I. The quantities Q and I are taken from reference 8; a is taken from reference 9; v is obtained from the elasticity constant; (c_{11}) given in reference 10; the mean line width for the nuclear resonance is taken as 10^4 sec^{-1}.[11] The quantity γ^2 is taken from the experimental data of Wikner and Das,[12] who computed the γ^2_T due to the quadrupole moment of the electronic shell of the atom. However, the theoretical relaxation times computed by these authors, T_T, do not coincide with the experimental values τ_E. According to the general theory given by Kranendonk[5] the spin-lattice relaxation time τ is proportional to γ^{-2}. It is reasonable to assume that γ^2_E, which takes account of all effects and agrees with experiment, will be given by $\gamma^2_E = \gamma^2_T \tau_T / \tau_E$.

In Table II are shown values of σ for transitions from the lower magnetic level with $m = I$.

4. CONCLUSION

Because of the lack of data, a comparison of theory with experiment is possible only for In^{115} in InSb.[4] The authors propagated ultrasound at a frequency $\nu = 9.976$ Mcs in a InSb single crystal located in magnetic fields $H_0 = 10.69 \times 10^3$ gauss and $H_0 = 5.35 \times 10^3$ gauss for the $\Delta m = 1$ and $\Delta m = 2$ transitions respectively. These authors observed clearly defined resonance absorption peaks and also investigated the dependence of these peaks on the angle θ between the magnetic field and the direction of propagation of the acoustic wave. The values $\theta = 0$ and π correspond to maxima for the $\Delta m = 1$ transitions and minima for the $\Delta m = 2$ transitions; the values $\theta = \pi/2$ and $3\pi/2$ correspond to maxima for $\Delta m = 2$ transitions and minima for $\Delta m = 1$ transitions. The resonance peaks are given in relative units. Minimum absorption corresponds to one relative unit σ_0, maximum corresponds to 7. The ratio

TABLE II

	$\sigma^{\perp}_{1z}\frac{T}{\nu^2}\cdot10^{17}$	$\sigma^{\perp}_{2x}\frac{T}{\nu^2}\cdot10^{17}$	$\sigma^{\perp}_{2y}\frac{T}{\nu^2}\cdot10^{17}$	$\sigma^{\parallel}_{1x}\frac{T}{\nu^2}\cdot10^{17}$	$\sigma^{\parallel}_{1y}\frac{T}{\nu^2}\cdot10^{17}$
Br^{79} in LiBr	0.295	0.64	0.295	0.295	0.295
Br^{79} in KBr	0.13	0.28	0.13	0.13	0.13
Br^{79} in AgBr	1.5	3.4	1.5	1.5	1.5
I^{127} in KI	11.3	12.7	5.7	11.3	11.3

of the absorption maxima for the transitions characterized by $\Delta m = 2$ and $\Delta m = 1$ is 0.66 ± 0.8.

To make a comparison with experiment we find the dependence of the absorption factor on the angle θ. Let the magnetic field be along the z axis and let the acoustic wave be propagated in the xz plane at an angle θ with respect to the z axis. Since σ is proportional to φ^2 all the $\sigma^{\perp}$ must be multiplied by $\sin^2\theta$ and the $\sigma^{\parallel}$ by $\cos^2\theta$. If we assume that waves of all three polarizations propagate, in the notation of Sec. 3

$$\sigma_1 = \sigma_{1x}^{\perp} + \sigma_{1y}^{\perp} + \sigma_{1z}^{\perp} + \sigma_{1x}^{\parallel} + \sigma_{1y}^{\parallel} + \sigma_{1z}^{\parallel}$$

$$= PD\xi^2(2\cos^2\theta + \sin^2\theta), \qquad \sigma_2 = \frac{13}{4} PD\eta^2 \sin^2\theta. \qquad (11)$$

Since P and D are positive, σ_1 has maxima for $\theta = 0$ and π and minima for $\theta = \pi/2,\ 3\pi/2$; σ_2 has maxima for $\theta = \pi/2,\ 3\pi/2$ and minima for $\theta = 0,\ \pi$, in accordance with experiment.

The theoretical ratio of the maximum values of σ_2 and σ_1 can be obtained taking σ_1 at $\theta = 0$ and σ_2 at $\theta = \pi/2$ in Eq. (11):

$$(\max \sigma_2 / \max \sigma_1)_T = 13\eta^2 / 4 \cdot 2 \cdot \xi^2 = 0.41.$$

It is noteworthy that of all the materials considered in Sec. 3 this ratio is less than unity only for InSb. The estimate was made out under the assumption that the linewidth is the same for $\Delta m = 1$ and $\Delta m = 2$. However, as is noted by the authors, the linewidth for the $\Delta m = 2$ signal is somewhat smaller. Consequently, with an exact calculation of the widths $(\max \sigma_2/\max \sigma_1)_T$ would increase.

At those values of θ for which there is theoretically no absorption, $\sigma = \sigma_0$.[4] If σ_0 refers to usual nonresonance absorption then it, and $\sigma_{1E} = 6\sigma_0$, can be estimated numerically from the dimensions of the crystal and the nonresonance acoustic absorption factor σ_{nonres}. Unfortunately the data on σ_{nonres} and on the velocity of sound and γ^2 required for a numerical calculation are not available in the literature. A rough estimate indicates that for agreement of $\max \sigma_{1E}$ and $\max \sigma_{1T}$ it is necessary to take $\gamma^2 > 2 \times 10^6$ at room temperature.

In conclusion we wish to express our gratitude to Prof. S. A. Al'tshuler for his valuable advice and interest in this work.

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PENETRATION OF AN ELECTROMAGNETIC FIELD INTO A PLASMA

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The depth of penetration of an electromagnetic field into a semi-infinite plasma in a magnetic field perpendicular to the plasma boundary is calculated.

SILIN[1] and Shafranov[2] have considered the penetration of an electromagnetic field into a uniform electron plasma which fills a half-space in the presence of an external magnetic field perpendicular to the plasma boundary, However, the ion motion, which is important at low frequencies, was not taken into account in references 1 and 2.

In the present work we determine the depth of penetration of circularly polarized electromagnetic waves for the case of perpendicular incidence with the ion motion taken into account. The plasma fills the half-space $z > 0$. The external magnetic field H_0 is perpendicular to the plasma boundary. It is assumed that the time dependence of all quantities can be written in the form $e^{-i\omega t}$ $(\mathrm{Im}\ \omega = -0)$. It is also assumed that the frequency ω is so large that "close" collisions can be neglected. In the case of specular reflection of the electrons and ions from the plasma boundary, using the kinetic equations for the electron and ion distribution functions and Maxwell's equations we find (cf. reference 1) for the electric field in the plasma:

$$E^{(\pm)}(z) = E_x(z) \pm iE_y(z) = -\frac{2cE^{(\pm)'}(0)}{\pi\omega}\int_0^\infty \frac{\cos(\omega n z/c)}{n^2 - \varepsilon^{(\pm)}(n)}\,dn, \qquad (1)$$

where

$$\varepsilon^{(\pm)}(n) = 1 + i\sqrt{\pi}\sum_{\alpha=e,i}\frac{\Omega_\alpha^2}{\omega^2}z_\alpha w(z_\alpha^{(\pm)}),$$

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right),$$

$$z_\alpha = (\sqrt{2}\beta_\alpha n)^{-1}, \quad z_\alpha^{(\pm)} = (1 \pm \omega_H^\alpha/\omega)/\sqrt{2}\beta_\alpha n,$$

$$\beta_\alpha = v_T^\alpha/c = (T_\alpha/m_\alpha c^2)^{1/2}, \quad \Omega_\alpha^2 = 4\pi e^2 n_0/m_\alpha,$$

$$\omega_H^\alpha = e_\alpha H_0/m_\alpha c.$$

When $\alpha = e$ the subscript refers to electrons, and when $\alpha = i$ to ions. The quantity T_α is the temperature of the gas of particles of type α, mass m_α and electric charge e_α $(e_i = e > 0)$, n_0 is the equilibrium electron density. $E^{(+)}(z)$ is the electric field associated with the extraordinary wave, $E^{(-)}(z)$ is the field of the ordinary wave.

In the general case, when spatial dispersion due to the thermal motion of the electrons and ions must be taken into account, Eq. (1) for $E^{(+)}(z)$ is extremely complicated. However, it is possible to find an asymptotic expression for $E^{(+)}(z)$ for large values of z which is determined essentially by the equilibrium distribution functions for the charged particles at high velocities v $(v \gg v_T$, where v_T is the average thermal velocity).

We denote the depth of penetration of the magnetic field into the plasma by the complex quantity $\delta_H^{(\pm)}$, which has the dimensions of length

$$\delta_H^{\pm} = \int_0^\infty [H_x(z) \pm iH_y(z)]\,dz/[H_x(0) \pm iH_y(0)]. \qquad (2)$$

Similar expressions are used for the depth of penetration of the electric field (E), the electron current (j_e), and the ion current (j_i). Except for the factor $i\omega/c$ the quantity in (2) is the surface impedance of the plasma. Using Eq. (1), from Maxwell's equations we have

$$\delta_H^{(\pm)} = \frac{2c}{\pi\omega}\int_0^\infty \frac{dn}{n^2 - \varepsilon^{(\pm)}(n)}, \qquad (3)$$

$$\delta_E^{(\pm)} = -(c^2/\omega^2\varepsilon^{(\pm)}(0))/\delta_H^{(\pm)}, \qquad (4)$$

$$\delta_{j_\alpha}^{(\pm)} = -\frac{i\sqrt{\pi}}{2}\frac{c}{\omega\varepsilon^{(\pm)}(0)}\left\{\int_0^\infty \frac{z_\alpha w(z_\alpha^{(\pm)})}{n^2 - \varepsilon^{(\pm)}(n)}\,dn\right\}^{-1}, \qquad (5)$$

where

$$\varepsilon^{(\pm)}(0) = \varepsilon^{(\pm)}(n)\big|_{n=0} = 1 - \Omega_e^2/\omega(\omega \mp |\omega_H^e|) - \Omega_i^2/\omega(\omega \pm \omega_H^i). \qquad (6)$$

We determine the quantities in Eqs. (3) — (5) for a number of limiting cases.

1. We assume that $|z_{e,i}^{(\pm)}| \gg 1$ for those values

of n which make an important contribution in the integrals (3), (5), (spatial dispersion is neglected). Then, from Eqs. (3) – (5), assuming that $\epsilon^{(\pm)}(n) \approx \epsilon^{(\pm)}(0)$, we have

$$\delta_H^{(\pm)} = \delta_E^{(\pm)} = \delta_{j_e}^{(\pm)} = \delta_{j_i}^{(\pm)} = \delta_0^{(\pm)} = ic/\omega\sqrt{\varepsilon^{(\pm)}(0)}. \quad (7)$$

In particular, at gyromagnetic resonances

$$\delta_0^{(+)} = (c/\Omega_e)(1 - |\omega_H^e|/\omega)^{1/2}, \qquad \omega \approx |\omega_H^e|,$$

$$\delta_0^{(-)} = (c/\Omega_i)(1 - \omega_H^i/\omega)^{1/2}, \qquad \omega \approx \omega_H^i.$$

Since $n \sim |\epsilon^{(\pm)}(0)|^{1/2}$ in Eqs. (3) and (5), the condition $|z^{(\pm)}| \gg 1$ is satisfied if

$$c/v_T^e\,|\varepsilon^{(\pm)}(0)|^{1/2} \gg 1. \quad (8)$$

The inequality in (8) is not satisfied for very large values of $|\epsilon^{(\pm)}(0)|$.

Below we consider the cases opposite to that given in (8). In these cases the depth of penetration of the electromagnetic field into the plasma is determined essentially by the thermal motion of the electrons and ions.

2. We assume that the chief contribution in the integral in (3) is due to those values of n for which $|z_e^{(\pm)}| \ll 1$. Then

$$\varepsilon^{(\pm)}(n) \approx i\sqrt{\pi/2}\,\Omega_e^2/\omega^2\beta_e n.$$

Whence it follows that in (3) the values $n \sim (\Omega_e^2/\omega^2\beta_e)^{1/3}$, are important. The condition $|z_e^{(\pm)}| \ll 1$ is satisfied if

$$(c\omega/v_T^e\Omega_e)\,|1 \mp |\omega_H^e|/\omega|^{1/3} \ll 1. \quad (9)$$

The depth of penetration of the magnetic field in this case is given by

$$\delta_H^{(\pm)} = \delta_a \equiv \frac{2}{3}\left(1 + \frac{i}{\sqrt{3}}\right)\left(\sqrt{\frac{2}{\pi}}\,\frac{c^2 v_T^e}{\Omega_e^2\omega}\right)^{1/3}. \quad (10)$$

A comparison of Eqs. (10) and (7) shows that because of (9) $|\delta_a| \gg |\delta_0^{(\pm)}|$. Equation (10) determines the surface impedance of the plasma when there is an anomalous skin effect (cf. also reference 1).

Furthermore, if (9) is satisfied we have

$$\delta_E^{(\pm)} = \frac{9}{8}\left(1 - \frac{i}{\sqrt{3}}\right)\left(\sqrt{\frac{\pi}{2}}\,\frac{c\omega}{v_T^e\Omega_e}\right)^{1/3} \times\left(1 \mp \frac{|\omega_H^e|}{\omega}\right)\left(1 \pm \frac{\omega_H^i}{\omega}\right)\frac{c}{\Omega_e}, \quad (11)$$

$$\delta_{j_e}^{(\pm)} = \frac{27}{16}\left(1 \mp \frac{|\omega_H^e|}{\omega}\right)\left(1 \pm \frac{\omega_H^i}{\omega}\right)\delta_a, \quad (12)$$

$$\delta_{j_i}^{(\pm)} = (v_T^i/v_T^e)\,\delta_{j_e}^{(\pm)} \quad \text{for } |\delta_a(\omega \pm \omega_H^i)| \ll v_T^i, \quad (13)$$

$$\delta_{j_i}^{(\pm)} = \delta_E^{(\pm)} \quad \text{for } |\delta_a(\omega \pm \omega_H^i)| \gg v_T^i. \quad (14)$$

It follows from Eqs. (9), (10) and (11) that $|\delta_E^{(\pm)}| \ll |\delta_0^{(\pm)}| \ll |\delta_a|$. In the case of an electronic gyromagnetic resonance $\omega \approx |\omega_H^e|$ and the depths of penetration of the electron current, the ion current, and the electric field of the extraordinary wave are reduced markedly.

3. Suppose that $\omega \ll |\omega_H^e|$. We assume that $|z_e^{(\pm)}| \gg 1$ and $|z_i^{(\pm)}| \ll 1$. Then

$$\varepsilon^{(\pm)}(n) \approx \pm\Omega_i^2/\omega\omega_H^i + i\sqrt{\pi/2}\,\Omega_i^2/\omega^2\beta_i n. \quad (15)$$

If the second term on the right side of Eq. (15) is much smaller than the first $\beta_i n \gg \omega_H^i/\omega$; then the important values in the integrals in Eqs. (3) and (5) are $n \sim \Omega_i/\sqrt{\omega\omega_H^i}$. In this case

$$\delta_H^{(\pm)} = (ic/\Omega_i)(\pm\omega_H^i/\omega)^{1/2}, \quad (16)$$

$$\delta_E^{(\pm)} = \delta_{j_e}^{(\pm)} = \pm(ic/\Omega_i)\,(1 \pm \omega_H^i/\omega)(\pm\omega_H^i/\omega)^{1/2}, \quad (17)$$

$$\delta_{j_i}^{(\pm)} = \mp i\sqrt{\pi/2}\,(v_T^i/\omega)(1 \pm \omega_H^i/\omega)/\varkappa^{(\pm)}, \quad (18)$$

where

$$\varkappa^{(+)} = \ln\alpha, \quad \varkappa^{(-)} = -\ln\alpha - i\pi/2,$$
$$\alpha = \sqrt{\pi/2}\,c\omega_H^i\sqrt{\omega_H^i}/v_T^i\Omega_i\sqrt{\omega}.$$

The condition $\beta_i n \gg \omega_H^i/\omega$ is satisfied if $\alpha \ll 1$. The inequalities $|z_e^{(\pm)}| \gg 1$ and $|z_i^{(\pm)}| \ll 1$ are satisfied when

$$\alpha m_i v_T^i/m_e v_T^e \gg 1, \quad \alpha\,|1 \pm \omega/\omega_H^i| \ll 1. \quad (19)$$

In the case of a gyromagnetic resonance $\omega \approx \omega_H^i$

$$\delta_H^{(-)} = c/\Omega_i, \quad \delta_E^{(-)} = (c/\Omega_i)(1 - \omega_H^i/\omega). \quad (20)$$

If, however $\alpha \gg 1$, when $\omega \ll |\omega_H^e|$, even in the absence of resonance $\omega \approx \omega_H^i$ we can use Eq. (7) for the ordinary wave. When $\omega \approx \omega_H^i$ we can neglect the first term in Eq. (15) as compared with the second (for the ordinary wave). In this case in the integrals in Eqs. (3) and (5) the values $n \sim (\Omega_i^2/\omega^2\beta_i)^{1/3}$ become important. The condition $|z_i^{(-)}| \ll 1$ is satisfied if, when $\alpha \gg 1$,

$$\alpha\,|\omega_H^i/\omega - 1|^{3/2} \ll 1. \quad (21)$$

In this case the depth of penetration is given by

$$\delta_H^{(-)} = \frac{2}{3}\left(1 + \frac{i}{\sqrt{3}}\right)\left(\sqrt{\frac{2}{\pi}}\,\frac{c^2 v_T^i}{\Omega_i^2\omega}\right)^{1/3},$$

$$\delta_E^{(-)} = \delta_{j_e}^{(-)} = \frac{c^2}{\Omega_i^2\,\delta_H^{(-)}}\left(1 - \frac{\omega_H^i}{\omega}\right),$$

$$\delta_{j_i}^{(-)} = -\frac{27}{16}\left(1 - \frac{\omega_H^i}{\omega}\right)\delta_H^{(-)}. \quad (22)$$

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THE RESONANCE OF CHARGE CARRIERS PRODUCED BY AN ULTRASONIC WAVE

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The interaction of charge carriers situated in a magnetic field with the electric field produced by an ultrasonic wave is considered. The power absorbed in unit volume is calculated for charges with scalar and tensor effective masses. The curve for absorbed energy has peaks where the ultrasonic frequency is $n\omega_0$ (n being an integer and ω_0 the cyclotron frequency of the carrier), provided the relaxation time $\tau \gg 1/\omega_0$. Because the ultrasonic wavelength is about 10^5 times smaller than that of light at the same frequency, polarization effects should be absent in experiments on ultrasonic resonance; this prevents the use of cyclotron resonance in semiconductors with a high concentration of free electrons.

THE experimental successes of the ultrasonic technique have made the problem of the interaction of charge carriers with sound waves one of current importance.

The transport of electrons by ultrasonic waves has been studied by Gurevich[1] and Parmenter.[2] In the present communication we consider the effect of ultrasonic waves on the electron gas of a semiconductor situated in a magnetic field.

We shall solve the problem in the classical approximation for which the de Broglie electron wavelength must be smaller than the sound wavelength:

$$\lambda_e \ll \lambda_s. \tag{1}$$

Using inequality (1) and the known values of the effective mass of the charge carriers and of the speed of sound c_0, one can derive a limiting temperature when the carriers are non-degenerate and a limiting concentration when they are degenerate; below these limits the formulae derived below are justified.

We use the method of the deformation potential[3,4] and write the interaction energy of an electron with an isotropically deformed crystal in the form

$$W = a(u_{xx} + u_{yy} + u_{zz}), \tag{2}$$

where a is the deformation potential and u_{ij} are the components of the deformation tensor. Then the force acting on an electron is

$$\mathbf{F} = \varkappa a u_{0yy} \cos(\omega t - \varkappa y + \beta_1), \tag{3}$$

if the deformation is caused by a monochromatic wave moving along the y axis. The magnetic field, H, is taken to be along the z axis. The radius vector of the electron $\mathbf{r}$ can be written as the sum of two components: the vector $\mathbf{R}$, referring the electron to the center of its orbit, and $\mathbf{r}_1$, the displacement of the center of the orbit in the time t from the instant of its last collision. The problem of finding $\mathbf{r}_1$ in the general case for the force (3) reduces to the solution of a non-linear differential equation. However, if

$$|\varkappa y_1| \ll 1, \tag{4}$$

the additional phase can be neglected, leaving only κR_y in (3). The value of y_1 can be obtained by considering the limiting case of a uniform electric field: $\mathbf{E} = \mathbf{E}_0 \cos \omega t$. Solving the kinetic equation with this force, we obtain for the magnitude of the displacement of the center of the orbit under resonance conditions

$$|y_{10}| = \frac{eE_0\tau}{m\omega_0}\sqrt{\frac{1+\tau^2\omega_0^2}{1+4\tau^2\omega_0^2}}, \qquad eE_0 = \varkappa a u_{0yy}. \tag{5}$$

From (5) it is seen that if $au_{0yy}\tau\omega_0/mc_0^2 \ll 1$, inequality (4) is satisfied. Not having data for comparison with experiment, we will assume that the parameters lie within the limits necessary for the latter inequality to be satisfied. As an example, it can be shown that if $c_0 = 5 \times 10^5$ cm/sec, $m = m_e$, $\tau\omega_0 = 5$, then au_{0yy} should be smaller than 10^{-4} ev.

1. THE POWER ABSORBED IN UNIT VOLUME

Because the field created by an ultrasonic wave is strongly inhomogeneous, instead of using the kinetic equation, as was done, for example, in references 5 and 6, we resort to a direct average of the energy absorbed by the individual particles.

In the interval of time Δt between two successive collisions, let $u(\beta_1, \beta, \Delta t, v_\perp)$ be the energy change of the charged particle with velocity component $v_\perp$ perpendicular to the magnetic field, with β the initial phase of the motion and with β_1 the phase of the force. After the time Δt, having changed its energy by u, the particle moves from the energy level $\epsilon - u$ to the level ϵ. In unit time and unit volume the number of such transitions will be

$$\frac{f(\varepsilon-u)\exp(-\Delta t/\tau)}{\Delta t}\frac{d\Delta t}{\tau}\frac{d\beta}{2\pi}\frac{d\beta_1}{2\pi}$$

$$\approx \frac{1}{4\pi^2}\left(f(\varepsilon)-u\frac{\partial f}{\partial \varepsilon}\right)\frac{\exp(-\Delta t/\tau)}{\Delta t\tau}\,d\beta_1 d\beta, \qquad (6)$$

where $f(\epsilon)$ is the distribution function normalized to the number of electrons in unit volume. Multiplying (6) by u and integrating with respect to β and β_1 from 0 to 2π, with respect to Δt from 0 to ∞, and over all the energy levels, we obtain the expression for the absorbed power U:

$$U = -\frac{1}{4\pi^2}$$

$$\times\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^{2\pi}\frac{u^2}{\Delta t}\frac{\partial f}{\partial \varepsilon}\exp(-\Delta t/\tau)\frac{d\Delta t}{\tau}\,d\varepsilon_\perp\, d\varepsilon_\parallel\, d\beta_1 d\beta. \qquad (7)$$

Here, $\epsilon_\perp$ is the orbital and $\epsilon_\parallel$ the residual part of the energy, and $\rho d\epsilon_\perp d\epsilon_\parallel$ is the number of states in the energy interval $d\epsilon_\perp d\epsilon_\parallel$.

According to (3), when (4) is satisfied:

$$u(\beta_1, \beta, \Delta t, v_\perp) = F_{0y}v_{0\perp}$$

$$\times\int_0^{\Delta t}\cos(\omega_0 t+\beta)\cos(\omega t-\varkappa R\sin(\omega_0 t+\beta)+\beta_1)\,dt. \qquad (8)$$

Substituting (8) in (7) and completing the integrations with respect to β_1, β, and Δt, we obtain

$$U = -\frac{4F_{0y}^2}{m}$$

$$\times\sum_{n=1}^{\infty} n^2\int\!\!\int\frac{J_n^2(z)}{z^2}\frac{\tau(1+\tau^2\omega_n^2+\tau^2\omega^2)}{(1+\tau^2\omega_n^2-\tau^2\omega^2)^2+4\tau^2\omega^2}\frac{\partial f}{\partial \varepsilon}\rho d\varepsilon_\perp d\varepsilon_\parallel, \qquad (9)$$

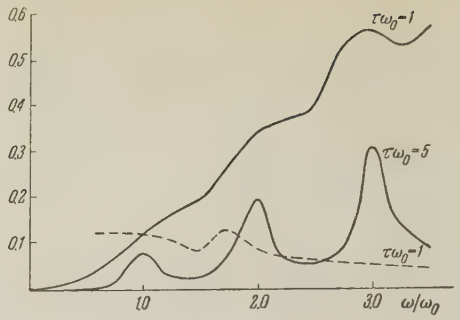
where $z = \kappa R$, $\omega_n = n\omega_0$, and J_n is the Bessel function of the n-th order.

As expected, in the limiting case in which $\kappa \to 0$, when $\tau = \text{const}$, $F_{0y} = \text{const}$, and classical statistics are used, (9) goes over to the well-known formula of the classical theory of cyclotron resonance (see reference 5).

To simplify calculations with Eq. (9), we will assume that $\tau = \text{const}$ and use classical statistics, After straightforward manipulation we find that

$$U = \frac{2a^2u_{0yy}^2\omega_0^2\tau N}{kT}\sum_{n=1}^{\infty}\frac{n^2(1+\tau^2\omega_n^2+\tau^2\omega^2)}{(1+\tau^2\omega_n^2-\tau^2\omega^2)^2+4\tau^2\omega^2}e^{-x}I_n(x). \qquad (10)$$

where I_n is the Bessel function of argument $x = (\omega/\omega_0)^2 kT/mc_0^2$.



In the figure, curves constructed with the aid of (10) are given for $\tau\omega_0 = 5$ and 1, and $kT/mc_0^2 = 0.5$; they show the relative amount of absorbed energy: $UkT/2a^2u_{0yy}^2\omega_0^2\tau N$, where N is the concentration of electrons.

For $\tau\omega_0 = 5$, resonance peaks of the first, second and higher orders are clearly visible. For $\tau\omega_0 = 1$, resonance maxima are observed starting with the third order. The increasing height of the peaks with increasing order is explained by the linear dependence of the force (3) on the frequency of the ultrasonic wave for constant deformation amplitude. The broken line is the curve for constant force amplitude. The more complicated nature of the absorption curve compared with cyclotron resonance is connected with the presence in (9) of the additional variable $z = \kappa R$.

2. ULTRASONIC RESONANCE FOR AN ELLIPSOIDAL SURFACE OF CONSTANT ENERGY

We will consider a hexagonal crystal and limit ourselves to the simple case in which the energy minimum is in the center of the Brillouin zone. The calculation for a more complicated surface can be carried out when the practical need arises.

The z axis is taken parallel to the principal symmetry axis of the crystal. The x and y axes are chosen so that κ is directed along the y axis. The magnetic field is taken along a line lying in the xz plane. The equations of motion of an electron in this case are (see reference 5):

$$\dot{p}_x = \omega_t\cos\vartheta p_y, \quad p_y = \omega_l\sin\vartheta p_z - \omega_t\cos\vartheta\, p_x,$$
$$\dot{p}_z = -\omega_t\sin\vartheta p_y, \quad \omega_{t,l} = eH/cm_{t,l}, \qquad (11)$$

The quantities m_t and m_l are determined from the formula for the energy, which, under the assumptions made above, must have the form

$$\varepsilon = (p_x^2+p_y^2)/2m_t + p_z^2/2m_l. \qquad (12)$$

Integrating (11), we find

$$x = R_x\cos(\omega_0 t+\beta)+V_{0x}t,$$
$$y = -R_y\sin(\omega_0 t+\beta),$$
$$z = R_z\cos(\omega_0 t+\beta)+V_{0z}t; \qquad (13)$$

$$R_x = (2\varepsilon_\perp / m_t)^{1/2}(\omega_t / \omega_0^2)\cos\vartheta,$$
$$R_y = (2\varepsilon_\perp / m_t)^{1/2}(1 / \omega_0),$$
$$R_z = (2m_t\varepsilon_\perp)^{1/2}(\omega_t / \omega_0^2)\sin\vartheta,$$
$$\omega_0^2 = \omega_t^2\cos^2\vartheta + \omega_l\omega_t\sin^2\vartheta. \quad (14)$$

We introduce the new variables

$$\xi_x = m_t^{-1/2}p_x, \quad \xi_y = m_t^{-1/2}p_y, \quad \xi_z = m_l^{-1/2}p_z. \quad (15)$$

In ξ phase space the vector $\xi_\parallel$ describing the residual motion of the center of the electron orbit is perpendicular to $\xi_\perp$ describing the orbital motion. Therefore, introducing a cylindrical system of co-ordinates with axis along $\xi_\parallel$, we obtain for the number of states in the energy interval $d\epsilon_\perp d\epsilon_\parallel$,

$$\rho d\varepsilon_\perp d\varepsilon_\parallel = 4\pi(m_t^2 m_l)^{1/2}\xi_\perp d\xi_\perp d\xi_\parallel / (2\pi\hbar)^3. \quad (16)$$

The force (3) in the case under consideration is

$$F = \varkappa u_{0yy}a_1\cos(\omega t - \varkappa R_y + \beta_1), \quad (17)$$

(here a_1 is one of the two unequal components of the deformation potential tensor: $a_{xx} = a_{yy} = a_1 \neq a_{zz} = a_2$)

$$u = \varkappa u_{0yy}a_1\omega_0 R_y \int_0^{\Delta t}\cos(\omega_0 t + \beta)\cos(\omega t - \varkappa R_y + \beta_1)\,dt. \quad (18)$$

After substituting for u, the integration of (7) with respect to angular variables leads to a formula differing from (9) only in the replacement of a^2/m by a_1^2/m_t.

Using (16) and integrating further with respect to $\xi_\perp$ and $\xi_\parallel$, under the same assumptions as were used to obtain (10), we arrive at an expression which can be obtained from (10) by substituting

$$\omega_0^2 = (eH / mc)^2 \to \omega_0^2 = \omega_t^2\cos^2\vartheta + \omega_t\omega_l\sin^2\vartheta,$$
$$x \to x_1 = (\omega / \omega_0)^2 kT / m_t c_{0\perp}^2,$$

where $c_{0\perp}$ is the velocity of sound in a direction perpendicular to the symmetry axis.

If the ultrasonic wave is directed along the z axis and the magnetic field along the x axis, we find $U \sim a_2^2$. If τ is weakly dependent on the direction of the magnetic field, then measurements of the heights of the resonance maxima for various directions of $\boldsymbol{\kappa}$ and $\mathbf{H}$ enable one to evaluate the relative magnitudes of the components of the deformation potential tensor.

In conclusion I regard it as a pleasant duty to express my gratitude to Yu. E. Perlin for useful discussion and valuable observations.

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THE MOTION OF AN ELECTRON IN A CRYSTAL LOCATED IN AN EXTERNAL FIELD

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Various forms of the equations of motion of an electron with an arbitrary dispersion law in a uniform magnetic and arbitrary electric fields are considered. A transition from the exact equation to approximate ones involves neglected values which are estimate. Special attention is paid to the nondiagonal terms (due to neighboring energy bands).

In a previous paper[1] the equation of motion of an electron with an arbitrary dispersion law in a magnetic field was obtained and solved. In the present paper this solution is extended to the case in which in addition to the uniform magnetic field $H = H_z$ there is also an arbitrary electrical field V in the crystal and, consequently, the eigenfunction of the electron satisfies equation:

$$-\frac{\hbar^2}{2m_0}\Delta\Psi - i\mu_0 Hy\frac{\partial\Psi}{\partial x} + \left(\frac{e^2H^2y^2}{2m_0c^2} + V_p + V - E\right)\Psi = 0 \quad (1)$$

(V_p is the periodic field of the crystal). In the absence of external fields ($V = 0$, $H = 0$) Eq. (1) has a known solution (s is the number of the energy band):

$$\psi_{\mathbf{k}s} = e^{i\mathbf{k}\mathbf{r}}u_s(\mathbf{k}, \mathbf{r}) = e^{i\mathbf{k}\mathbf{r}}\sum_{\mathbf{h}} b_{\mathbf{h}s}(\mathbf{k})\, e^{2\pi i\mathbf{h}\mathbf{r}}; \quad (2)$$

$$E = E_s(\mathbf{k}) = \sum_{\mathbf{n}} A_{\mathbf{n}s}\, e^{i\mathbf{k}\mathbf{n}}. \quad (3)$$

For crystals with a center of symmetry the coefficients $b_{\mathbf{h}s}(\mathbf{k})$ in the expansion of the Bloch function (2) are real, even functions of arguments $\mathbf{k} + 2\pi\mathbf{h}$. They satisfy relation:

$$\sum_{\mathbf{h}} b_{\mathbf{h}s}(k)\, b_{\mathbf{h}s'}(\mathbf{k}) = \delta_{ss'}. \quad (4)$$

Any sum over all $\mathbf{h}$ of the form $\Sigma f(\mathbf{k} + 2\pi\mathbf{h})$ is a periodic function of the wave vector $\mathbf{k}$ and can be written in the form completely similar to (3). Taking advantage of this, let us introduce the notation

$$\sum_{\mathbf{g}} b_{\mathbf{g}s}(\mathbf{k})\frac{\partial^{l_1+l_2+l_3} b_{\mathbf{h}+\mathbf{g},\,s'}}{\partial k_1^{l_1}\partial k_2^{l_2}\partial k_3^{l_3}} = \sum_{\mathbf{m}} B^{l_1l_2l_3}_{\mathbf{m},\,\mathbf{h},\,s,\,s'}\, e^{i\mathbf{k}\mathbf{m}};\quad B^{\mathbf{l}}_{\mathbf{m}0ss'} \equiv B^{\mathbf{l}}_{\mathbf{m}ss'}; \quad (5)$$

$$\sum_{\mathbf{g}} b_{\mathbf{g}s}(\mathbf{k})\frac{\partial^{l_1+l_2+l_3} b_{\mathbf{h}+\mathbf{g},\,s'}}{\partial k_1^{l_1}\partial k_2^{l_2}\partial k_3^{l_3}}(k_2 + 2\pi g_2) = \sum_{\mathbf{m}} C^{l_1l_2l_3}_{\mathbf{m},\,\mathbf{h},\,s,\,s'}\, e^{i\mathbf{k}\mathbf{m}}. \quad (6)$$

Generally speaking, in order of magnitude,

$$B^{l_1l_2l_3} \sim a^{l_1+l_2+l_3}, \qquad C^{l_1l_2l_3} \sim a^{l_1+l_2+l_3-1} \quad (7)$$

(a is the lattice constant), with the exception of the special cases examined below.

In the following discussion the properties of the $B_{\mathbf{m}}$ coefficients, expressed by formulas (8) through (10), are important:

$$B^{100}_{\mathbf{m}ss} = B^{010}_{\mathbf{m}ss} = B^{001}_{\mathbf{m}ss} = 0. \quad (8)$$

When $s \neq s'$ these coefficients are different from zero [therefore they are the principal representatives of the neighboring bands, i.e., of the nondiagonal terms of Eq. (1)] and have the property:

$$B^{100}_{\mathbf{m}ss'} = -B^{100}_{\mathbf{m}s's} \quad \text{etc.} \quad (9)$$

Notice also that

$$B^{000}_{\mathbf{m}ss'} = \delta_{ss'}\delta_{\mathbf{m}0}. \quad (10)$$

We seek the solution of (1) in the form of expansion:

$$\Psi = \sum_s \int g_s(\mathbf{k})\, e^{i\mathbf{k}\mathbf{r}} u_s(\mathbf{k} + \mathbf{i}y/\alpha_0^2,\ \mathbf{r})\, d\mathbf{k}, \quad (11)$$

$\mathbf{i}$ is the unit vector of the X axis, $\alpha_0 = \sqrt{\hbar c/eH}$ is a quantity having the meaning of the least "radius of rotation" of a free electron in a magnetic field. For fields $H \ll 10^9$ oersteds, $\alpha_0 \gg a$ (a is the lattice constant). With a field $H = 10^4$ oersteds, $\epsilon = a/\alpha_0 \approx 10^{-2}$).

The equation of electron motion in $\mathbf{k}$ space, i.e., the equation for $g_s(\mathbf{k})$, will be obtained by substituting (11) in (1), multiplying by the integrand function $e^{i\mathbf{k}'\mathbf{r}}u_{s'}(\mathbf{k}' + \mathbf{i}y/\alpha_0^2,\ \mathbf{r})$, and integrating over the entire space.

In reference 1 we retained only the first two terms of the expansion with respect to parameter $\epsilon^2 = a^2/\alpha_0^2$ (ϵ^0 and ϵ^2); here we retain terms $\sim\epsilon^0$, ϵ^2, and ϵ^4, and neglect only terms $\sim\epsilon^6$ and exponentially small ones of the type $\exp(-1/\epsilon^2)$. For this reason the equation will be cumbersome;

however, simpler forms will be used below, and in each case we shall be able to assess the error resulting from ignoring the small terms. Terms of the order ϵ^6 or $\exp(-1/\epsilon^2)$ (recalling that usually $\epsilon \sim 10^{-2}$) will hardly be necessary, and thus the equation written below can be considered as accurate as equation (1):

$$\sum_{\mathbf{n}} A_{\mathbf{ns}} \exp\left\{ i\mathbf{kn} - i\frac{n_1 n_2}{2\alpha_0^2} \right\} g_s\left(k_2 - \frac{n_1}{\alpha_0^2}\right)$$
$$- \frac{1}{\alpha_0^2} \sum_{\mathbf{m},\, s' \neq s} e^{i\mathbf{km}} \left\{ B^{010}_{\mathbf{mss'}} \sum_{\mathbf{n}} A_{\mathbf{ns'}}(m_1 + n_1) \right.$$
$$\times \exp\left\{ i\mathbf{kn} - i\frac{n_1 n_2}{\alpha_0^2} \right\} g_{s'}\left(k_2 - \frac{n_1 + m_1}{\alpha_0^2}\right)$$
$$\left. + \left(\frac{i\hbar^2}{m_0} C^{100}_{\mathbf{mss'}} - m_1 E B^{010}_{\mathbf{mss'}} \right) g_{s'}\left(k_2 - \frac{m_1}{\alpha_0^2}\right) \right\} + \frac{1}{2\alpha_0^4} \sum_{s'} H^{(2)}_{ss'} g_{s'}$$
$$+ \sum_{\substack{\mathbf{ms'} \\ \mathbf{l},\, \mathbf{h}}} B^{\mathbf{l}}_{\mathbf{m},\, \mathbf{h},\, s,\, s'} \frac{e^{i\mathbf{km}}}{l_1!\, l_2!\, l_3!} \int g_{s'}(\mathbf{k}') V\left(\mathbf{k}' - \mathbf{k} + \mathbf{j}\frac{m_1}{\alpha_0^2} + 2\pi\mathbf{h}\right)$$
$$\times (k_1' - k_1)^{l_1} (k_2' - k_2)^{l_2} (k_3' - k_3)^{l_3} d\mathbf{k}' = E g_s(\mathbf{k}). \qquad \text{(I)}$$

We have introduced here the symbols

$$V(\varkappa) = \int e^{i\varkappa \mathbf{r}} V(\mathbf{r})\, d\tau, \qquad (12)$$

$\mathbf{j}$ is the unit vector of the y axis, and $g(k_1, k_2 - k_2^0, k_3)$ is understood for $g(k_2 - k_2^0)$. An explicit expression for the term of the order α_0^{-4} is written out in the Appendix.

The "magnetic terms" of Eq. (I) contain no components with $\mathbf{h} \neq 0$, because they are exponentially small [$\sim \exp(-1/\epsilon^2)$]. Let us explain the cases in which we can drop these components in the last term connected with the electrical field $V(\mathbf{r})$, of the left part of (I).

This can be done, for example, in the particular case when $V(\mathbf{r})$ reduces to a polynomial (and, in addition, $H = 0$), since then (12) will be expressed in terms of $\delta(\mathbf{k}' - \mathbf{k} + 2\pi\mathbf{h})$ and its derivatives, while the point $\mathbf{k}^1 = \mathbf{k} - 2\pi\mathbf{h}$ lies outside the elementary $\mathbf{k}$-space cell over which the integration is performed. If $V(\mathbf{r})$ is an analytic function having no poles with $V(\infty) = 0$, and if the distance at which it essentially disappears is equal to A, then the integral (12) for $\mathbf{h} \neq 0$ will be of the order $e^{-A/a}$, or even $\exp\{-(A/a)^2\}$. Therefore when $A \gg a$ it is possible to drop from (I) terms with $\mathbf{h} \neq 0$ as exponentially small. If however $V(\mathbf{r})$ has a pole (for example, $V \sim 1/r$) or merely a discontinuity of the derivative at any point (for example, $e^{-\alpha r}$), the terms with $h \neq 0$, generally speaking, cannot be dropped. In these cases an additional investigation of (I) is necessary.

Let us now examine the problem of the influence of neighboring bands. It is best to examine first the nondiagonal "electrical" terms. Two obvious questions arise: 1) at which $V(\mathbf{r})$ are the nondiagonal terms most substantial, and 2) which features of the neighboring bands are capable of making their contribution to (I) more substantial.

The answer to the first question comes from the fact that, according to (9), $B^{000}_{ss'} = \delta_{ss'}$; i.e., when $s = s'$, there are no multipliers $(k_1' - k_1)^{l_1}$ etc. in the integrand of (I). Therefore the contribution of neighboring bands will be small if $V(\mathbf{r})$ is a smooth function, varying little over the lattice constant. If $V(\mathbf{r})$ changes substantially over the distance A, and $A \gg a$, then the contribution of neighboring bands can be written as a series in powers of the small parameter, a/A (beginning, generally speaking, with the first power of this parameter, since $B^{100}_{ss'} \neq 0$). We assume, of course, that coefficients $B^{100}_{ss'}$ (meaning also $B^{010}_{ss'}$ and $B^{001}_{ss'}$) are of the order a [see Eq. (7)].

The answer to the second question is exactly connected with the possible deviations from the relation $B^{100}_{ss'} \sim a$. We recall that these values are determined by formula (5), which in the given case has the form

$$\sum_{\mathbf{m}} B^{100}_{\mathbf{mss'}} e^{i\mathbf{km}} = \sum_{\mathbf{h}} b_{\mathbf{hs}}\, \partial b_{\mathbf{hs'}} / \partial k_1.$$

Thus the coefficients B^{100} can be very large if derivatives $\partial b_{\mathbf{hs'}}/\partial k_i$ are large. If we express $E_s(\mathbf{k})$ by $b_{\mathbf{hs}}$ using formula $E_s = \int \psi^*_{\mathbf{ks}} H \psi_{\mathbf{ks}} d\mathbf{t}$ and expand E_s is powers of $(k_1 - k^0)^2$ near the point of effective mass, the derivatives $\partial b_{\mathbf{hs}}/\partial k_i$ and $\partial^2 b_{\mathbf{hs}}/\partial k_i^2$ will enter the coefficients of $(k_i - k_i^0)^2$. Therefore, generally speaking, the larger these derivatives, the smaller will be the effective mass. Thus, we can expect anomalously large values of $B^{100}_{ss'}$ in cases in which effective mass m^* in the neighboring bands is anomalously small (as was pointed out in another way by Adams[2]), or in general the coefficients $b_{\mathbf{hs}}$, and consequently the energy, have an anomalously rapid variation with change in the wave vector. Thus, for example, in the approximation of weakly bound electrons, for values of $\mathbf{k}$ close to the boundaries of the band, we have $\partial b_{\mathbf{h}}/\partial k_1 \sim a(\hbar^2/m_0 a^2 V_g)$, which can also be written as $\partial b_{\mathbf{hs}}/\partial k_1 \sim a m_0/m^*$. Here V_g is the Fourier coefficient of the potential, and in this approximation $V_g \ll \hbar^2/m_0 a^2$, and therefore $\partial b/\partial k$ is not of the order a, but is significantly greater. The coefficients $B^{100}_{ss'}$ increase correspondingly, and along with them the contribution of the neighboring bands.

The diagonal coefficient B^{200}_{ss} can also become anomalously large for the same reason, but the contribution of values $B^{\mathbf{l}}$ with large $|\mathbf{l}|$ is gen-

erally less, because of the factors $(k_i' - k_i)^l i$ in the integrand.

The influence of the neighboring bands in the "magnetic terms" of Eq. (I) is usually weaker and is easier to evaluate, since there is always a small multiplier α_0^{-2} standing before each nondiagonal magnetic term. Therefore their contribution cannot be greater than $a\alpha_0^{-2}B^{010}$, while at the same time the contribution of the neighboring bands in the "electrical terms" is of the order B^{100}/A. Here A is the distance at which $V(\mathbf{r})$ changes significantly,* $\alpha_0/a \sim 10^2$ and $\alpha_0 \sim 2 \times 10^{-6}$ cm with fields $H \sim 10^4$ oersteds, so that $\alpha_0^2/a \sim 10^{-4}$ cm. If $A \ll \alpha_0^2/a$, the effect of neighboring bands in the "electrical" terms is more substantial than in the "magnetic" ones.

If the contribution of neighboring bands can be considered as a perturbation, it will be less than the foregoing estimates, because the product of rapidly oscillating (for levels not too low) and noncoincident functions g_S and $g_{S'}$ will enter into the perturbation matrix element. In the case in which the electrical field is absent,[1] this gives, in addition to multiplier α_0^{-2}, another multiplier $\epsilon = a/\alpha_0$, or even an exponentially small one.†

In the case when $V(\mathbf{r})$ changes so slowly that we can drop in Eq. (I) terms with $\mathbf{h} \neq 0$ and the nondiagonal terms describing the neighboring bands, Eq. (I) assumes the simple form

$$\sum_{\mathbf{n}} A_{\mathbf{n}s} \exp\{i\mathbf{k}\mathbf{n} - in_1n_2/2\alpha_0^2\}\, g_s\left(k_1,\ k_2 - \frac{n_1}{\alpha_0^2},\ k_3\right) + \int g_s(\mathbf{k}')\, V(\mathbf{k}' - \mathbf{k})\, d\mathbf{k}' = E g_s(\mathbf{k}). \qquad \text{(II)}$$

If we set here $H = 0$, i.e., $\alpha_0^2 = \infty$, the first term on the left will simply take the form $E_S(\mathbf{k})\, g_S(\mathbf{k})$, while Eq. (II) changes into the Slater equation[3] written in the $\mathbf{k}$ representation. A particular consequence is the effective-mass method. The Vannier function method used by Slater and several other authors makes it difficult to evaluate errors which occur in going from the accurate equation (I) to the "quasi-classical" equation (II).

Equation (II) is of the integro-difference type. In a number of cases it can be written also in a differential-difference form. This includes, in particular, the case when $V(\mathbf{r})$ is a polynomial. For each term of the polynomial $x^n y^m z^l$ there is a corresponding Fourier component

$$(x^n y^m z^l)_{\mathbf{k}'-\mathbf{k}} = i^{-(n+m+l)}\, \delta^{(n)}_{(k_1'-k_1)}\, \delta^{(m)}_{(k_2'-k_2)}\, \delta^{(l)}_{(k_3'-k_3)}$$

[the δ-function is taken to mean everywhere $\int e^{i\mathbf{k}\mathbf{r}} d\tau$ without the multiplier $(2\pi)^{-3}$]. After integrating with $g(\mathbf{k})$ in (II) this term takes the form

$$i^{n+m+l}\, \partial^{n+m+l} g(\mathbf{k}) / \partial k_1^n\, \partial k_2^m \partial k_3^l.$$

Consequently the integral in (II) can be written in this case in the form of a differential operator $V(\hat{x}, \hat{y}, \hat{z})$, where $\hat{x} = i\partial/\partial k_1$, $\hat{y} = i\partial/\partial k_2$, $\hat{z} = i\partial/\partial k_3$. Equation (II) takes the form

$$\sum_{\mathbf{n}} A_{\mathbf{n}s} \exp\{i\mathbf{k}\mathbf{n} - in_1n_2/2\alpha_0^2\}\, g_s(k_1, k_2 - n_1/\alpha_0^2,\ k_3) + V(i\partial/\partial k_1,\ i\partial/\partial k_2, i\partial/\partial k_3) g_s = E g_s(\mathbf{k}). \qquad \text{(III)}$$

Equation (II) can be written in the same form also in the general case in which integration with respect to $\mathbf{k}$ in (II) can be extended over the entire $\mathbf{k}$ space (which corresponds to replacing the quasi-momentum with the momentum).

The left part of (III) can be represented completely in operator form, if we account for the fact that, as shown in reference 4, the first sum in (III) (under the condition of using the operation of total symmetrization[4]) can be written as $E_S(\hat{\kappa}_1, k_2, k_3)$, where $\hat{\kappa}_1 = k_1 - (1/i\alpha_0^2)\, \partial/\partial k_2$:

$$E_s\left(k_1 - \frac{1}{i\alpha_0^2}\frac{\partial}{\partial k_2},\ k_2,\ k_3\right) g_s + V\left(i\frac{\partial}{\partial \mathbf{k}}\right) g_s = E g_s. \qquad \text{(IV)}$$

As an example, let us examine the motion of an electron in crossed electric ($V = Fy$) and magnetic fields. We first neglect the nondiagonal terms, thereby obtaining the equation

$$\sum_{\mathbf{n}} A_{\mathbf{n}s} \exp\left\{i\mathbf{k}\mathbf{n} - i\frac{n_1n_2}{2\alpha_0^2}\right\} g_s\left(k_2 - \frac{n_1}{\alpha_0^2}\right) + iF\frac{\partial g_s}{\partial k_2} = E g_s(\mathbf{k}). \qquad (13)$$

As shown in reference 5, the quasi-classical solution of equation (13) does not differ at all in principle from the solution of the equation when $F = 0$. The electron in this case moves in $\mathbf{k}$-space along a trajectory shifted in the direction perpendicular to both fields. It is easy to show that nondiagonal terms will give a correction of the order ϵ^3 or less (see footnote*) compared with the corresponding diagonal ones.

As another example, let us examine the motion of an electron in a field $V = Fy$ without a magnetic field. Neglecting nondiagonal terms, we obtain from (13):

$$E_s(\mathbf{k})\, g_s(\mathbf{k}) + iF\partial g_s/\partial k_2 = E g_s(\mathbf{k}). \qquad (14)$$

A solution of this equation is

$$g_s(\mathbf{k}) = \exp\left\{\frac{1}{iF}\int [E - E_s(\mathbf{k})]\, dk_2\right\}. \qquad (15)$$

For a free particle, when $E(\mathbf{k}) = \hbar^2k^2/2m$, we obtain the $\mathbf{k}$-representation Airy function.[6] With an

*The distance at which $V(A) \sim E_s(\mathbf{k}) - E_{s'}(\mathbf{k})$ (see the last of the examples given below).

†Depending on whether the $\mathbf{k}$ trajectories are intersected in zones s and s'. "Trajectory" means the intersection between a surface $E(\mathbf{k}) = \text{const}$ and a plane perpendicular to the magnetic field.

arbitrary periodic $E_s(\mathbf{k})$ we obtain the $\mathbf{k}$-representation of a function which could be called the "Airy crystal function." We recall that the eigenfunction in ordinary space is expressed by formula (11), which in the given case has the form $\Psi_s = \int g_s(\mathbf{k})\psi_{ks}d\mathbf{k}$. If the field F is weak, this integral can be computed by the steepest-descent method. Since ψ_{ks} contains a multiplier e^{ik_2y}, then Ψ oscillates in the interval where the derivative with respect to k_2 of

$$k_2y - \frac{1}{F}\int_0^{k_2}[E - E_s]\,dk_2$$

vanishes. From this we immediately get a number of known facts concerning the motion of an electron in a uniform electric field. The motion of a free electron is bounded on one side, and the motion of an electron in a band is bounded on two sides by the limits $y_{max} = (E - E_{s\,min})/F$, $y_{min} = (E - E_{s\,max})/F$. Outside this interval, Ψ diminishes exponentially.

Now let us examine the motion of an electron in a field $V = Fy$, with allowance for the nondiagonal terms (assuming them to anomalously large). It is sufficient to examine two bands, s and r. Equation (I) in this case will be written in the form of the system

$$\begin{aligned} iF\partial g_s/\partial k_2 + (E_s - E)\,g_s + iFBg_r = 0,\\ iF\partial g_r/\partial k_2 + (E_r - E)\,g_r - iFBg_s = 0. \end{aligned} \qquad (16)$$

Here the property (9) is accounted for and the symbol $B = \sum B^{010}_{msr}e^{ikm}$ is introduced. By substituting

$$g_s = \exp\left\{\frac{1}{iF}\int\left[E - \frac{1}{2}(E_s + E_r)\right]dk_2\right\}\phi_s,$$
$$g_r = \exp\left\{\frac{1}{iF}\int\left[E - \frac{1}{2}(E_s + E_r)\right]dk_2\right\}\phi_r$$

we bring system (16) to a more symmetrical form

$$\begin{aligned} iF\phi_s' + (E_s - E_r)\,\phi_s + iFB\phi_r = 0,\\ iF\phi_r' - (E_s - E_r)\,\phi_r - iFB\phi_s = 0. \end{aligned} \qquad (17)$$

Assuming the field F to be weak, we seek a quasiclassical solution. We find that if we write $\psi_s = \exp(i\Phi_1/F)$, the solution for the second band has the form $\psi_r = iF\Phi_2\exp(i\Phi_1/F)$, where Φ_1 and Φ_2 can be expanded by powers of iF.

$$\Phi_1 = \frac{1}{2}\int(E_s - E_r)\,dk_2 + F^2\int\frac{B^2}{E_s - E_r}\,dk_2 + \ldots,$$
$$\Phi_2 = -\frac{B}{E_s - E_r} - iF\left(\frac{B}{E_s - E_r}\right)'\frac{1}{E_s - E_r} + \ldots \qquad (18)$$

In this manner, the solution has the following final form:

$$g_s = \exp\left\{\frac{1}{iF}\int_0^{k_2}(E - E_s)\,dk_2 + iF\int_0^{k_2}\frac{B^2}{E_s - E_r}\,dk_2 + \ldots\right\}$$

$$g_r = -\frac{iFB}{E_s - E_r}$$
$$\times\exp\left\{\frac{1}{iF}\int_0^{k_2}(E - E_s)\,dk_2 + iF\int_0^{k_2}\frac{B^2}{E_s - E_r}\,dk_2 + \ldots\right\} \qquad (19)$$

In these formulas we can interchange s and r and thereby obtain a second solution.

As expected, the weaker the field F, the less the significance of the second band. Its contribution is the more significant the larger B is and the stronger the bands overlap, i.e., the smaller the difference $E_s - E_r$ (for a given $\mathbf{k}$). If the difference $E_s(\mathbf{k}) - E_r(\mathbf{k})$ is very small in any region of $\mathbf{k}$ values, while the value of B is large, the contribution of both bands may become commensurate.

APPENDIX

Terms of the order $1/\alpha_0^4$ in Eq. (I) have the following explicit form:

$$\frac{1}{2\alpha_0^4}\sum_{s'}H_{ss'}g_{s'} = \frac{1}{2\alpha_0^4}\Bigg\{\sum_{m,\,s'}B^{020}_{mss'}e^{ikm}\sum_n A_{ns'}e^{ikn}(m_1 + n_1)^2 g_{s'}$$
$$\times\left(k_2 - \frac{m_1 + n_1}{\alpha_0^2}\right) - E\sum_{m,\,s'}B^{020}_{mss'}e^{ikm}g_{s'}\left(k_2 - \frac{m_1}{\alpha_0^2}\right)$$
$$+ \frac{2i\hbar^2}{m_0}\sum_{m,\,s'}(B^{100}_{mss'} + C^{110}_{mss'})\,e^{ikm}g_{s'}\left(k_2 - \frac{m_1}{\alpha_0^2}\right)$$
$$+ \frac{1}{4}\sum_n A_{ns}e^{ikn}n_1^2n_2^2\,g_s\left(k_2 - \frac{n_1}{\alpha_0^2}\right)$$
$$- \frac{\hbar^2}{m_0}\sum_{m,\,s'}B^{200}_{mss'}e^{ikm}g_{s'}\left(k_2 - \frac{m_1}{\alpha_0^2}\right)\Bigg\}.$$

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ON CAUSALITY IN A THEORY WITH AN INDEFINITE METRIC

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The possibility of constructing a unitary and macroscopically causal scattering matrix in a theory with an indefinite metric is examined. The construction is carried out in the framework of perturbation theory by means of the interaction Lagrangians of the complete (physical plus sum of nonphysical) fields. By a special choice of the spectrum of the nonphysical fields it is possible to satisfy the requirements of unitarity and macroscopic causality in the second and third orders. These requirements cannot, however, be fulfilled together in the fourth order; thus within the framework of our assumptions it is not possible to construct a unitary and macroscopically causal scattering matrix in a theory with indefinite metric.

1. Recently much attention has been given to the ideas of Heisenberg, who has proposed the introduction of an indefinite metric in the space of the state amplitudes, for the purpose of eliminating divergences. This program leads to the appearance of a number of difficulties associated with the necessity of introducing "nonphysical" fields that have negative norms, and with violation of the condition that the scattering matrix be unitary. Several schemes have been proposed, however, that make it possible to eliminate the "nonphysical" states from the asymptotic expressions for observable quantities and to restore the unitary character of the scattering matrix.

The papers in question do not deal with the problems of causality, which certainly are of great interest. In the present paper we examine the question of the possibility of constructing a macroscopically causal theory with indefinite metric, on the basis of rather general postulates.

Following Heisenberg, we shall assume that the complete space H of state amplitudes decomposes into two orthogonal subspaces: H_1, the subspace of physical state amplitudes, and H_2, the subspace of "nonphysical" state amplitudes. As is done in reference 2, we introduce the operator P that projects the complete state amplitude onto the subspace H_1 of physical state amplitudes and is such that $P = P^+ = P^2$. We represent the complete field $\chi(x)$ in the form

$$\chi(x) = \varphi_0(x) + \sum_n c_n \varphi_{m_n}(x), \tag{1}$$

where $\varphi_0(x)$ is the physical field of mass m and $\varphi_{m_n}(x)$ is a "nonphysical" field of mass m_n, and impose the following commutation conditions:

$$\{\varphi_0(x),\ \varphi_0(y)\} = \mathcal{D}(x-y),$$

$$\{\varphi_{m_n}(x),\ \varphi_{m_n}(y)\} = \varepsilon_n \mathcal{D}_{m_n}(x-y),$$

where $\epsilon_n = \pm 1$. In accordance with this the bracket for the complete field $\chi(x)$ is

$$\{\chi(x),\ \chi(y)\} = D(x-y) = \mathcal{D}(x-y) + \Delta(x-y), \tag{2}$$

where

$$\Delta(x-y) = \sum_n \varepsilon_n c_n^2 \mathcal{D}_{m_n}(x-y).$$

In what follows we shall use the term "nonphysical bracket" to denote $\Delta(x-y)$. By a suitable choice of ϵ_n, c_n, and m_n we can always secure regularity of the function $D(x-y)$ on the light cone.

In references 1 — 3 the theory with indefinite metric has been constructed by the use of the concept of the scattering matrix in the complete space H. As has been shown in reference 4, within the framework of perturbation theory it is impossible to have any theory of such a sort that satisfies the condition of macroscopic causality. In the present paper we give up the idea of the scattering matrix in the complete space. Assuming, however, that the interaction occurs between the complete fields, we introduce the concept of the interaction Lagrangian of the complete fields.

2. The problem is to construct a scattering matrix $\tilde{S}$ connecting the asymptotic expressions for the physical state amplitudes of the subspace H_1, and which should be expansible in a series in the interaction constant

$$\tilde{S} = 1 + \sum_n \tilde{S}_n.$$

We shall use the hypothesis of adiabatic turning on and off of the interaction. Then $\tilde{S}$ is a functional of the "turning-on intensity" $g(x)$.[5]

We impose the following conditions on $\tilde{S}$:

1) Relativistic covariance.

2) Unitarity: $\tilde{S}\tilde{S}^+ = 1$.

3) Weak causality. We formulate this condition in the following way: suppose there are two non-overlapping space-time regions G_1 and G_2 in which the interaction is turned on with the respective intensities $g_1(x)$ and $g_2(y)$, and such that $x \lesssim y$. Then the difference

$$R_{12} = \tilde{S}(g_1 + g_2) - \tilde{S}(g_2)\tilde{S}(g_1)$$

must go to zero with sufficient rapidity as the "distance" between the regions increases; the necessary degree of rapidity of this approach to zero will be discussed later.

4) Furthermore we require that $\tilde{S}_n$ be a polynomial function of the complete-field Lagrangians and operators P; we emphasize that in the present paper we are using perturbation theory, and therefore we require that all the above conditions be fulfilled in each order independently.

It is easy to show that an $\tilde{S}_n$ satisfying the first and fourth conditions is a polynomial function of operators P and S_i, where

$$1 + S_1 + S_2 + \ldots = T\exp\left\{i\int L(x)\,g(x)\,dx\right\} = S(g).$$

The matrix $S(g)$ has the form of the usual scattering matrix in the complete space. Without assigning to it any meaning, we shall hereafter use only its formal properties of unitarity, $SS^+ = 1$, and strict causality[5]

$$S(g_1 + g_2) = S(g_2)\,S(g_1).$$

3. We now proceed to the direct construction of a matrix $\tilde{S}$ that satisfies the conditions listed above.

We shall first examine S correct to the second order. The most general form for $(\tilde{S})_2$ is

$$(\tilde{S})_2 = P(1 + S_1 + aS_2 + a_1S_1S_1 + a_2S_1PS_1)\,P. \qquad (3)$$

The coefficients of the first two terms are chosen from considerations of correspondence with classical theory.

From the unitarity condition we have

$$a = a^*, \qquad a_1^* + a_1 + 1 = 0, \qquad a_2^* + a_2 - 1 = 0. \qquad (4)$$

In order to make use of the causality condition we consider the expression

$$(R_{12})_2 = P[(a + a_1)\,S_1(g_2)\,S_1(g_1) + (a_2 - 1)\,S_1(g_2)\,PS_1(g_1)$$
$$+ a_1S_1(g_1)\,S_1(g_2) + a_2S_1(g_1)\,PS_1(g_2)]\,P. \qquad (5)$$

We must require that this quantity go to zero with increase of the "distance" between the regions G_1 and G_2. Let us expand this expression by Wick's theorem and examine the terms not containing contractions. These terms do not depend on the "distance" between the regions, and therefore their sum must be identically zero. From this consideration we get

$$a + 2a_1 + 2a_2 - 1 = 0. \qquad (6)$$

Then from Eqs. (4) and (6) we have

$$a = 1, \qquad a_1 = -{}^1/_2 + i\alpha, \qquad a_2 = -(-{}^1/_2 + i\alpha),$$

where α is an arbitrary real number. Therefore the right member of Eq. (5) can be rewritten in the form

$$P[(-{}^1/_2 + i\alpha)\,S_1(g_2)\;(1 - P)\,S_1(g_1)$$
$$+ ({}^1/_2 + i\alpha)\,S_1(g_1)\,(1 - P)\,S_1(g_2)]\,P. \qquad (7)$$

This expression can be handled conveniently by means of the following lemma, which will be proved in the Appendix.

Suppose that in the expression

$$P\Pi(P,\ \chi_1(x_{k_1})\ldots\chi_n(x_{k_n}),\ \chi_1(y_{l_1})\ldots\chi_m(y_{l_m}))\,P \qquad (*)$$

Π is a polynomial function of the operators P and χ, the x_{k_j} being points of the region G_1 and the y_{1_j} points of the region G_2. Then the necessary and sufficient condition for the expression (*) to be equal to a sum of terms each proportional to a "nonphysical contraction" depending on $(x_{k_j} - y_{1_j})$ is that it be possible to put (*) in the form of a sum of terms of the type

$$P\Pi_1(P, \ldots\chi_i(x_{k_i})\ldots\chi_j(y_{l_j})\ldots)$$
$$\times P\Pi_2(P, \ldots\chi_i(x_{k_i})\ldots)(1 - P)\;P\Pi_3(P, \ldots\chi_j(y_{l_j})\ldots)$$
$$\times P\Pi_4(P, \ldots\chi_i(x_{k_i})\ldots\chi_j(y_{l_j})\ldots)\,P. \qquad (8)$$

Returning to the expression (7), we see that according to the lemma it is equal to a sum of terms each proportional to a "nonphysical contraction" of fields from the regions G_1 and G_2. Thus a sufficient condition for causality in second order is that "nonphysical contractions" approach zero rapidly with increase of the "distance" between the regions.

On the other hand, if in the expansion of the expression (7) by Wick's theorem we take the terms that depend linearly on contractions, which by the lemma are necessarily "nonphysical," we can note

that they are of different operator structures. And since only the contractions depend on the "distance" between the regions, the expression (7) will go to zero with increase of the "distance" only if the "nonphysical contractions" go to zero.

Thus the necessary and sufficient condition for the fulfillment of the causality condition for S in second order is that the "nonphysical contractions" of fields from the regions G_1 and G_2 go to zero with increase of the distance between these regions.

Let us examine the behavior of the contractions $\mathcal{D}(x-y)$ with increase of the "distance" between x and y. For simplicity we confine ourselves to scalar fields. Three cases can arise: 1) $\lambda < 0$, 2) $\lambda > 0$, 3) $\lambda = 0$, where $\lambda = (x^0-y^0)^2 - (\mathbf{x}-\mathbf{y})^2$. From the explicit form of the contractions[5] we have in the first case the following asymptotic expression:

$$\mathcal{D}(x-y) \sim \sqrt{m}(-\lambda)^{3/4} \exp\{-m\sqrt{-\lambda}\}.$$

It is seen that for spacelike intervals $\mathcal{D}(x-y)$ falls off exponentially. In the second case the asymptotic expression has the form

$$\mathcal{D}(x-y) \sim \sqrt{m}\,\lambda^{-3/4} \exp\{\pm im\sqrt{\lambda}\},$$

i.e., for timelike intervals $\mathcal{D}(x-y)$ oscillates (the gradual falling off does not prevent the propagation of particles through macroscopic times).

As was first shown by Fierz[6] in a discussion of the first type of regular theory proposed by Heisenberg,[1] precisely this fact leads to the violation of macroscopic causality in second order, owing to the propagation of "nonphysical" particles through macroscopic times.

We can, however, try to change the asymptotic behavior of the "nonphysical contractions" and thus remove this objection. In fact, if instead of the discrete spectrum (1) we introduce a spectrum in which each "nonphysical" field is averaged over a suitably small range of masses with some weight factor, for example a Gaussian distribution, the contraction of a "nonphysical" field takes the form

$$\mathcal{D}_{m_n} = \int \frac{1}{M} \exp\left\{-\frac{(m-m_n)^2}{M^2}\right\} \mathcal{D}_m(x-y)\,dm,$$

and the asymptotic expression for the "nonphysical contraction" gets an exponentially decreasing factor because of the averaging of rapidly oscillating functions:

$$\Delta(x-y) \sim \sum_n \lambda^{-3/4}\sqrt{m_n} \times \exp\{\pm im_n\sqrt{\lambda}\} \exp\left\{-\lambda\frac{M^2}{4}\right\}. \quad (9)$$

At the same time the regular behavior of the function $D(x-y)$ on the light cone is not destroyed.

Thus by the use of a "smeared-out" spectrum of the "nonphysical" fields the causality condition can be fulfilled in second order also for timelike intervals.*

As has been shown by B. V. Medvedev (private communication), the violation of causality in second order in the case considered by Fierz is easy to understand if we go back to the connection of the theory with indefinite metric with nonlocal theory. In fact, in the theory with indefinite metric one is actually using a Pauli-Villars regularization with finite masses

$$\frac{1}{m^2-p^2-i\varepsilon} - \frac{1}{m_1^2-p^2-i\varepsilon} = \frac{1}{m^2-p^2-i\varepsilon}\,\frac{m_1^2-m^2}{m_1^2-p^2-i\varepsilon}$$

(for simplicity we consider one "nonphysical" field). The cut-off factor $(m_1^2-m^2)/(m_1^2-p^2-i\varepsilon)$ can be referred not to the propagation function but to the vertex. Such a situation corresponds to a nonlocal theory with a factorizable form factor. The violation of causality in the case of a discrete spectrum of "nonphysical" fields is due to the fact that for the resulting form factor the Chretien-Peierls conditions[7] are not fulfilled, because such a form factor has a pole on the real axis of the squared momentum. Our way of introducing the "nonphysical" fields also receives a natural interpretation, since we in fact achieve the removal of the singularity by integrating a generalized function in a class of sufficiently smooth functions,[5] i.e., we achieve the fulfillment of the Chretien-Peierls conditions.

Thus by choice of the coefficients a, a_1, a_2 and of a special form of the spectrum of the "nonphysical" fields one can satisfy the conditions of unitarity and causality of $\tilde{S}$ in second order. We have finally for $(\tilde{S})_2$ the expression

$$(\tilde{S})_2 = P[1 + S_1 + S_2 + (-1/2 + i\alpha) S_1 (1-P) S_1] P, \quad (10)$$

which contains one arbitrary real parameter α.

4. Before going on to further orders, we note that:

a) the contractions $\mathcal{D}(x-y)$ of the physical fields fall off slowly for timelike intervals (we cannot require fast falling off, since then there would be no propagation of physical particles through macroscopic times),

*We do not examine here the fulfillment of the causality condition for regions lying along the light cone. It can, however, be shown, by use of the Chretien-Peierls method,[7] that the causality condition can be fulfilled in this case also, if we introduce the "nonphysical" fields in the way that has been described. This result is not needed in our further argument.

b) the "nonphysical contractions" fall off exponentially both for spacelike and also for timelike intervals,

c) the lemma formulated above is valid. Therefore a necessary and sufficient condition for the fulfillment of the causality condition in the higher orders is that the differences $(R_{12})_n$ be sums of terms of the form (8).

The possibility of choosing only such terms leads in the third order of perturbation theory to the following expression for $(\tilde{S})_3$:

$$(\tilde{S})_3 = P\{(\tilde{S})_2 + S_3 + (-{}^1/_2 + i\alpha)[S_1(1-P)S_2 + S_2(1-P)S_1 - S_1(1-P)S_1(1-P)S_1]\}P. \quad (11)$$

It can be checked by direct verification that the expression (11) is unitary. We note that $(\tilde{S})_3$ depends on the parameter α introduced in the second order; i.e., in the third order the degree of arbitrariness has not increased.

5. Finally, let us examine $\tilde{S}$ in fourth order. Here, as in the third order, the causality condition demands that $(R_{12})_4$ be a sum of terms of the type (8); after simple but very cumbersome calculations this condition gives

$$\begin{aligned}(\tilde{S})_4 = P\{&(\tilde{S})_3 + S_4 + (-{}^1/_2 + i\alpha)[S_1(1-P)S_3 \\ &+ S_2(1-P)S_2 + S_3(1-P)S_1 - S_1(1-P)S_1(1-P)S_2 \\ &- S_1(1-P)S_2(1-P)S_1 - S_2(1-P)S_1(1-P)S_1 \\ &+ S_1(1-P)S_1S_1(1-P)S_1] \\ &+ bS_1(1-P)S_1PS_1(1-P)S_1\}P, \end{aligned} \quad (12)$$

where in order to satisfy the requirements of causality we must have

$$b = {}^1/_4 - \alpha^2 - i\alpha. \quad (13)$$

For $(\tilde{S})_4$ the requirement of unitarity reduces to

$$b^* + b + \alpha^2 - {}^3/_4 = 0. \quad (14)$$

Comparing Eq. (13) with Eq. (14), we get $\alpha^2 + \frac{1}{4} = 0$; that is, the parameter α, which was introduced in the second order as a real quantity on the basis of the requirement of unitarity, must be purely imaginary.

Thus we arrive at an obvious contradiction: the conditions of unitarity and causality in fourth order are in conflict with the condition of unitarity in second order.

To get a clearer idea of the nature of the violation of causality, let us impose on the coefficient b only the condition (14) obtained from the unitarity condition for $(\tilde{S})_4$, but not the condition (13), and examine the structure of the terms that violate causality. These terms in $(S)_4$ are

$$\begin{aligned}P\{&({}^1/_2\alpha^2 + {}^1/_8 + i\alpha + i\beta)\, S_1(g_1)(1-P) \\ &\times S_1(g_1)PS_1(g_2)(1-P)S_1(g_2) \\ &+ (-{}^1/_2\alpha^2 - {}^1/_8 + i\alpha + i\beta)\, S_1(g_2)(1-P) \\ &\times S_1(g_2)PS_1(g_1)(1-P)S_1(g_1)\}P, \end{aligned} \quad (15)$$

where β is an arbitrary real number.

From the structure of the expression (15) it is clear that it can be put in the form of a sum of terms each proportional to a "nonphysical contraction," but these contractions depend on the "distance" between points in the same region (G_1 or G_2). Since of course "distances" between points in the same region can be arbitrarily small, for the fulfillment of the condition of causality we have to require that the "nonphysical contractions" vanish for arbitrary "distances". In this case we practically arrive at the ordinary theory, since the contractions of the complete fields reduce to those of the physical fields, and as a result of the action of the operator P normal products of complete field operators are equal to normal products of the corresponding physical fields (see Appendix).

On the other hand, if we set $\alpha = -\beta$, then the expression (15) reduces to a sum of terms each proportional to a contraction of the physical fields, depending on the "distance" between points of the regions G_1 and G_2. By introducing a "smeared-out" spectrum of physical fields we can, of course, arrange matters so that their contractions fall off sufficiently rapidly with increase of the "distance" between the regions. But, as was noted above, this leads to the impossibility of the propagation of physical particles through macroscopic times.

6. Thus we have shown that even the weak causality condition is incompatible with the unitarity condition for the scattering matrix $\tilde{S}$ connecting the asymptotic state amplitudes of the physical subspace H_1, if it is constructed by means of the interaction Lagrangians of the complete fields.

This result is not unexpected, because also in nonlocal theory, with which the theory with indefinite metric is closely related, it has not been possible to reconcile the conditions of unitarity and causality.[8]

Naturally the question of the construction of a theory with indefinite metric without use of the idea of the complete-field Lagrangaian is at present still an open one.

In conclusion we emphasize once again that this discussion has all been within the framework of perturbation theory. Therefore we have not touched at all on the question of the possibility of compensation between violations of causality and unitarity in different orders. This extremely in-

teresting possibility remains open as before. Unfortunately the realization of such a program, like every attempt at any considerable advance beyond the framework of perturbation theory, encounters very serious difficulties.

We take occasion to express our deep gratitude to B. V. Medvedev for his constant interest in this work and a number of helpful suggestions. We also express our gratitude to N. N. Bogolyubov, D. V. Shirkov, and M. K. Polivanov for a helpful discussion.

APPENDIX

Proof of the lemma. It is easy to convince oneself that

$$P\varphi_0^{(+)} = \varphi_0^{(+)}P, \qquad P\varphi_0^{(-)} = \varphi_0^{(-)}P,$$
$$P\varphi_n^{(+)} = 0, \qquad \varphi_n^{(-)}P = 0.$$

Here $\varphi_0^{(+)}$ and $\varphi_0^{(-)}$ are the positive- and negative-frequency parts of the physical field, and $\varphi_n^{(+)}$ and $\varphi_n^{(-)}$ have the same meaning for a "nonphysical" field. The signs of the frequencies have the same meanings as in reference 5. Then the action of the operator P on a normal product of complete-field operators χ is:

$$P : \chi_1(x_{k_1}) \ldots \chi_n(x_{k_n}) : P$$
$$= P : \varphi_{01}(x_{k_1}) \ldots \varphi_{0n}(x_{k_n}) : P. \qquad (A.1)$$

Proof of the sufficient condition. Using Wick's theorem, we can put Π_2 and Π_3 in the form

$$\Pi_2 = \sum A(x_{k_1}, \ldots, x_{k_n}) : \ldots \chi_i(x_{k_i}) \ldots :,$$

$$\Pi_3 = \sum B(y_{l_1}, \ldots, y_{l_m}) : \ldots \chi_j(y_{l_j}) \ldots :,$$

where A and B are coefficient functions. Furthermore,

$$P\Pi_2 P = \sum A(x_{k_1}, \ldots, x_{k_n}) : \ldots \varphi_{0i}(x_{k_i}) \ldots :,$$

$$\Pi_2\Pi_3 = \sum\sum A(x_{k_1}, \ldots, x_{k_n})$$
$$\times B(y_{l_1}, \ldots, y_{l_m}) : \ldots \chi_i(x_{k_i}) \ldots : : \ldots \chi_j(y_{l_j}) \ldots :$$
$$= \sum\sum A(x_{k_1}, \ldots, x_{k_n}) B(y_{l_1}, \ldots, y_{l_m})$$
$$\times \sum K(x_k - y_l) : \ldots \chi_i(x_{k_i}) \ldots \chi_j(y_{l_j}) \ldots :,$$

where $K(x_k - y_l)$ is the product of just those contractions of fields in which a field from Π_2 is contracted with a field from Π_3. Then

$$P\Pi_2\Pi_3 P = \sum\sum A(x_{k_1}, \ldots, x_{k_n}) B(y_{l_1}, \ldots, y_{l_m})$$
$$\times \sum K(x_k - y_l) : \ldots \varphi_{0i}(x_{k_i}) \ldots \varphi_{0j}(y_{l_j}) \ldots : \qquad (A.2)$$

On the other hand

$$P\Pi_2 P\Pi_3 P = \sum\sum A(x_{k_1}, \ldots, x_{k_n}) B(y_{l_1}, \ldots, y_{l_m})$$
$$\times \sum K'(x_k - y_l) : \ldots \varphi_{0i}(x_{k_i}) \ldots \varphi_{0j}(y_{l_j}) \ldots : \qquad (A.3)$$

where A and B are the same in Eqs. (A.2) and (A.3), and K′ differs from K only in that all contractions of complete fields are replaced by contractions of physical fields. Subtracting Eq. (A.3) from Eq. (A.2), we have

$$P\Pi_2(1-P)\Pi_3 P = \sum\sum A(x_{k_1}, \ldots, x_{k_n}) B(y_{l_1}, \ldots, y_{l_m})$$
$$\times \sum [K(x_k - y_l) - K'(x_k - y_l)] :$$
$$\times \ldots \varphi_{0i}(x_{k_i}) \ldots \varphi_{0j}(y_{l_j}) \ldots : \qquad (A.4)$$

Breaking each complete contraction in K up into a sum of contractions of physical and "nonphysical" fields and subtracting K′, we find that each term of the expression (A.4) is either equal to zero or proportional to a "nonphysical contraction," as was to be proved.

Proof of the necessary condition. Suppose that

$$P\Pi P = \sum \Delta(x_k - y_l) P\Pi' P, \qquad (A.5)$$

but

$$\Delta(x_k - y_l) P = P\chi(x_k)(1-P)\chi(y_l) P. \qquad (A.6)$$

Substituting (A.6) into Eq. (A.5), we have

$$P\Pi P = \sum P\chi(x_k)(1-P)\chi(y_l) P\Pi' P, \qquad (A.7)$$

as was to be proved.

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ANGULAR CORRELATIONS NEAR MULTIPLE-PRODUCTION THRESHOLDS

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Limiting angular correlations near threshold are found for reactions that have as their final products an infinitely heavy nucleus and two, three, or four identical fermions with spin $\frac{1}{2}$.

IN the treatment of reactions near threshold it is usually assumed that the final products of a reaction are in s states, since the contributions from non-zero orbital angular momenta can be neglected. If there are among the final products $N > 2\sigma + 1$ identical fermions (σ is the spin of these fermions), then they cannot all be in s states; even at threshold wave functions satisfying the Pauli principle must in this case contain nonvanishing orbital angular momenta. Inclusion of effects of the Pauli principle leads to the appearance of angular correlations and to a change of the energy-dependences of cross-sections near threshold.[1]

For uncharged products the limiting angular correlations (output channel energy $E \to 0$) caused by exchange symmetry can be calculated in a number of cases independently of the concrete mechanism of the reaction. This can be done in cases in which because of different energy dependences only one channel with a definite symmetry can contribute at the threshold. For charged products all channels (with different orbital angular momenta, with different symmetries) have the same energy dependence at threshold, and to get the limiting angular distributions one must know the weights of the various channels. This requires further study of the concrete mechanism of the reaction.

We here calculate the angular correlations for reactions that have as their final stages the emission of N uncharged fermions with spin $\frac{1}{2}$ by an infinitely heavy nucleus ($N = 2, 3, 4$). In this case a channel with definite symmetry corresponds to a definite total spin S. The dependence of the reaction amplitude (in the momentum representation) on the momenta of the emerging particles is given near threshold by functions of the form

$$f(\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N) = A_\alpha (k_{\alpha_1})_i (k_{\alpha_2})_k \ldots (k_{\alpha_m})_l T_{ik\ldots l}, \quad (1)$$

which are the first terms of the expansion of the exact amplitude in powers of the momenta. Here $(\mathbf{k}_\alpha)_i \equiv k_{\alpha i}$ is the i-th component of the momentum of the α-th particle, A_α is symmetrization by the Young diagram corresponding to the given channel, and $T_{ik\ldots l}$ is a certain tensor of the m-th rank that is independent of the momenta and contains the dependence of the amplitude on directions in space that are privileged for the given reaction (the rank m is the same as the lowest degree in the expansion of the amplitude in powers of the momenta[1]). In the general case one can construct for each channel several such functions differing only in the numbering of the particles. These functions (and their linear combinations) form the basis of a certain irreducible representation of the permutation group, of dimensionality $r \geq 1$ (cf., e.g., reference 2).

Analogously, a certain representation of the same dimensionality (that associated with the first representation) is given by spin functions of the form

$$\chi(\sigma_1, \sigma_2 \ldots \sigma_N) = A^*_\alpha \prod_\alpha \chi(\sigma_\alpha), \quad (2)$$

where $\chi(\sigma_\alpha)$ are spin functions of the α-th particle (hereafter we shall write $\chi(\sigma_\alpha = \frac{1}{2}) = \xi_\alpha$, $\chi(\sigma_\alpha = -\frac{1}{2}) = \eta_\alpha$), and A^*_α is symmetrization by the associated (transposed) Young diagram, which is obtained from the diagram for the coordinate functions by interchange of rows and columns.

The cross section of the reaction is determined by the square of the "complete" amplitude, which is a certain bilinear combination of the coordinate and spin functions:

$$F(\mathbf{k}_1 \ldots \mathbf{k}_N, \sigma_1 \ldots \sigma_N) = \sum_{i,k=1}^{r} c_{ik} f_i \chi_k,$$

$$\sigma_N = \int |F|^2 \prod_\alpha k_\alpha^2 dk_\alpha d\Omega_\alpha \delta(k^2 - 2mE)$$

$$= E^{(3N-2)/2+m} \int \rho_N(\theta_1, \varphi_1 \ldots \theta_N, \varphi_N) \prod_\alpha d\Omega_\alpha. \quad (3)$$

The correlation function $\rho_N(\vartheta_i) = \overline{\rho_N(\theta_1\varphi_1 \ldots \theta_N\varphi_N)}$ is obtained by averaging the absolute square of the amplitude (3) over all directions in space and integrating over the possible distributions of the energy among the particles; it depends only on the angles ϑ_i between the various pairs of particles ($i = 1, 2, \ldots\ldots N(N-1)/2$). We emphasize that the meaning of the amplitude for a given channel (definite N, S, S_Z) belongs only to the "complete" amplitude F, which involves all possible coordinate functions f_i. If, for example, we are interested only in the energy dependence of the cross-section, there is no need to construct the amplitude (3), since actually all we need to calculate is the degree m, which is the same for all the f_i. For the calculation of the angular distribution it is already essential to know the weights with which the various coordinate functions occur in the amplitude (3).

From the mathematical point of view the construction of the amplitude (3) means separation of the antisymmetric representation of the permutation group from the direct product of the two representations given by the coordinate and spin functions.[2,3] We note that although the construction of the coordinate and spin bases is not an unambiguous operation, all bases lead to the same amplitude. The reaction amplitude has been found in the following way: an orthonormal spin basis and the corresponding representation were constructed, then a coordinate basis was chosen to give the associated representation, and finally the antisymmetric function was separated out from the direct product of the two bases.

1. N = 2. The only contribution at threshold is that from the channel with S = 0, since the channel with S = 1 is suppressed by an extra power of the energy E in the cross section. The spin and coordinate functions

$$\chi = (\xi_1\eta_2 - \eta_1\xi_2)/\sqrt{2}, \qquad f = \text{const}$$

give respectively an antisymmetric and a symmetric representation. The amplitude $F = f\chi$ gives $\rho_2(S=0) = \text{const}$. If in virtue of some selection rule the channel with S = 0 is closed (for example, if the total angular momentum in the final state must be different from zero), then the reaction goes in the channel with S = 1, for which

$$\chi = \xi_1\xi_2, \qquad f = (k_1 - k_2)_i T_i,$$
$$F = f\chi, \quad \rho_2(S=1) \sim \pi - {}^8/_3 \cos\vartheta.$$

We have written out only the spin function with the maximum value of S_Z, since the channels with different S_Z do not interfere and give the same angular dependences.

2. N = 3. The reaction goes in the channel with $S = \frac{1}{2}$, for which the two bases (spin and coordinate)

$$\chi_1 = (\xi_1\eta_2 - \eta_1\xi_2)\xi_3/\sqrt{2},$$
$$\chi_2 = [2\xi_1\xi_2\eta_3 - (\xi_1\eta_2 + \eta_1\xi_2)\xi_3]/\sqrt{6},$$
$$f_1 = (k_1 - k_2)_i T_i, \qquad f_2 = (k_1 + k_2 - 2k_3)_i T_i/\sqrt{3}$$

give the same two-dimensional representation, with the following matrices for the interchanges of pairs of particles:

$$M_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M_{13} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$
$$M_{23} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

The amplitude and the correlation function have the forms

$$F = f_1\chi_2 - f_2\chi_1, \qquad \rho_3(S = {}^1/_2) \sim \pi - \frac{8}{9}\sum_i \cos\vartheta_i.$$

For the total spin $S = {}^3/_2$ the representations are one-dimensional:

$$\chi = \xi_1\xi_2\xi_3,$$
$$f = [(k_1 - k_2)_i(k_2 - k_3)_k - (k_2 - k_3)_i(k_1 - k_2)_k]T_{ik}, \quad F = f\chi$$
$$\rho_3(S = \tfrac{3}{2}) \sim (\pi/2)\left(1 - \tfrac{1}{3}\sum_i \cos^2\vartheta_i\right)$$
$$+ \tfrac{8}{9}\sum_{i\neq k}\cos\vartheta_i\cos\vartheta_k - \tfrac{8}{9}\sum_i \cos\vartheta_i.$$

3. N = 4. The channels with S = 0 and S = 1 have the same energy dependence $\sigma_4 \sim E$.[7]

a) S = 0. The spin and coordinate bases

$$\chi_1 = (\xi_1\eta_2 - \eta_1\xi_2)(\xi_3\eta_4 - \eta_3\xi_4)/2,$$
$$\chi_2 = \{2\xi_1\xi_2\eta_3\eta_4 + 2\eta_1\eta_2\xi_3\xi_4 - (\xi_1\eta_2 + \eta_1\xi_2)(\xi_3\eta_4 + \eta_3\xi_4)\}/2\sqrt{3}$$
$$f_1 = \{(k_1 - k_2)_i(k_3 - k_4)_k + (k_3 - k_4)_i(k_1 - k_2)_k\}T_{ik},$$
$$f_2 = \{2k_{1i}k_{2k} + 2k_{2i}k_{1k} + 2k_{3i}k_{4k} + 2k_{4i}k_{3k}$$
$$- (k_1 + k_2)_i(k_3 + k_4)_k - (k_3 + k_4)_i(k_1 + k_2)_k\}T_{ik}/\sqrt{3}$$

give the same two-dimensional representation, with the matrices

$$M_{12} = M_{34} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{13} = M_{24} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix},$$
$$M_{23} = M_{14} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

The amplitude and the correlation function have the forms

$$F = f_1\chi_2 - f_2\chi_1,$$
$$\rho_4(S=0) \sim \frac{\pi}{8}\left(3 + \sum_i \cos^2\vartheta_i\right) - \frac{1}{3}\sum_{i\neq k}\cos\vartheta_i\cos\vartheta_k$$
$$+ \frac{8+\pi}{3\pi}\sum{}'\cos\vartheta_i\cos\vartheta_k - \frac{1}{3}\sum_i \cos\vartheta_i,$$

where

$$\sum{}' \cos\vartheta_i\cos\vartheta_k=\cos\vartheta_{12}\cos\vartheta_{34}$$
$$+\cos\vartheta_{13}\cos\vartheta_{21}+\cos\vartheta_{14}\cos\vartheta_{23}.$$

b) S = 1. The spin basis

$$\chi_1=\xi_1\xi_2(\xi_3\eta_4-\eta_3\xi_4)/\sqrt{2},\qquad \chi_2=(\xi_1\eta_2-\eta_1\xi_2)\xi_3\xi_4/\sqrt{2},$$
$$\chi_3={}^1/_2\{(\xi_1\eta_2+\eta_1\xi_2)\xi_3\xi_4-\xi_1\xi_2(\xi_3\eta_4+\eta_3\xi_4)\}$$

gives a three-dimensional representation, with the matrices

$$M_{12}=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix},\qquad M_{34}=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

$$M_{13}=\frac{1}{2}\begin{pmatrix}1&1&-\sqrt{2}\\1&1&\sqrt{2}\\-\sqrt{2}&\sqrt{2}&0\end{pmatrix},\qquad M_{14}=\frac{1}{2}\begin{pmatrix}1&-1&\sqrt{2}\\-1&1&\sqrt{2}\\\sqrt{2}&\sqrt{2}&0\end{pmatrix},$$

$$M_{23}=\frac{1}{2}\begin{pmatrix}1&-1&-\sqrt{2}\\-1&1&-\sqrt{2}\\-\sqrt{2}&-\sqrt{2}&0\end{pmatrix},$$

$$M_{24}=\frac{1}{2}\begin{pmatrix}1&1&\sqrt{2}\\1&1&-\sqrt{2}\\\sqrt{2}&-\sqrt{2}&0\end{pmatrix}. \qquad (4)$$

The coordinate basis

$$f_1=\{(k_1-k_2)_i(k_1+k_2-k_3-k_4)_k$$
$$-(k_1+k_2-k_3-k_4)_i(k_1-k_2)_k\}T_{ik},$$
$$f_2=\{(k_1+k_2)_i(k_3-k_4)_k-(k_3-k_4)_i(k_1+k_2)_k$$
$$+2k_{3i}k_{4k}-2k_{4i}k_{3k}\}T_{ik},$$
$$f_3=\sqrt{2}\{(k_1-k_2)_i(k_3-k_4)_k-(k_3-k_4)_i(k_1-k_2)_k\}T_{ik}$$

gives the other three-dimensional representation, with matrices that are the same as the matrices (4) except for a factor -1. The amplitude and correlation function have the forms

$$F=f_1\chi_1+f_2\chi_2+f_3\chi_3,$$
$$\rho_4(S=1)\sim\frac{1}{16}\pi\left(6-\sum_i\cos^2\vartheta_i\right)+\frac{1}{6}\sum_{i\neq k}\cos\vartheta_i\cos\vartheta_k$$
$$-\frac{1}{6}\sum{}'\cos\vartheta_i\cos\vartheta_k-\frac{1}{3}\sum_i\cos\vartheta_i.$$

The amplitudes that have been found can be used for the construction of angular dependences in reactions with charged products, since although the ratio of the contributions of successive channels, $\sim(Ze^2mR_0)^2$, does not vanish for $E\to 0$, it can still be small. Unlike the case of uncharged particles, the angular part does not depend on the distribution of the energy among the particles (instead of the momenta we must insert the unit vectors $\mathbf{k}_\alpha/k_\alpha$), and therefore the correlation function is the same as the mean square of the amplitude.

In conclusion we emphasize that the basic condition for the applicability of the method used here and in reference 1 is the existence of a finite range of the reaction (the radius R_0). In this case the formulas we have obtained are the first terms of the expansions of the exact values (or their asymptotic expressions) in powers of $\xi=(2mE)^{1/2}R_0$ (cf. also reference 4). Besides this one assumes a sufficiently smooth dependence of the wave function on the energy near threshold, i.e., the absence of resonance effects in the region $\xi\leq 1$. This condition can be violated, for example, if the entire system or some subsystem of the final products has an energy level (actual or virtual) near the threshold. Thus for two nucleons there exists near zero energy the virtual level $\epsilon=0.07$ Mev (for S = 0), and therefore for reactions with nucleons in the final state the range of validity of the threshold relations is further restricted to small values of the ratio E/ϵ. In the range $E/\epsilon\lesssim 1$ it is already necessary to take into account effects caused by the interaction of pairs of nucleons.[5,6] Another example of a distortion of the threshold relations is the case in which the region $\xi\lesssim 1$ includes the threshold of another reaction (for two final products this effect has been studied in references 7 and 8).

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ON THE DOPPLER EFFECT IN AN ANISOTROPIC AND GYROTROPIC MEDIUM

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The Doppler effect is treated for motion of an oscillator along the axis of a gyrotropic anisotropic crystal. General formulas for the energy of the radiation are obtained; these can be used, in particular, to obtain formulas for the Cerenkov radiation of a charge and of a dipole. A number of properties of the radiation that are peculiar to an anisotropic, gyrotropic medium are investigated.

THE Doppler effect has great importance in many fields of application. Hitherto only the case of motion of a source in an isotropic medium has been studied; it has been shown that in the presence of dispersion, splitting of the Doppler frequency occurs (the complex Doppler effect[1,2]). In connection with numerous investigations of the ionosphere (for example, with the aid of high-speed earth satellites) and of an electronic plasma, which in the presence of a magnetic field has the properties of an anisotropic and gyrotropic crystal, it is of interest to investigate the properties of the field of a radiator moving in such a medium. In this case there arise new effects, manifesting themselves in changes of the components of the electromagnetic field and in an additional splitting of the original frequency, dependent on the degree of anisotropy and on the value of the gyration parameter.

In the present article, in order to elucidate the principal peculiarities of the phenomenon, we consider the problem of the field of an electric oscillator of arbitrary orientation, moving along the axis of an anisotropic and gyrotropic crystal.

1. Let a point oscillator be moving with constant velocity $\mathbf{v}$ along the z axis, which coincides with the crystal axis; let the oscillator have, in its proper system of coordinates, frequency ω_0', electric moment $\mathbf{p}_0' \cos \omega_0' t$, and magnetic moment $\mathbf{m}_0' \cos \omega_0' t$. Just as in reference 1, we replace the moving oscillator by a system of "equivalent" electric and magnetic harmonic oscillators of density

$$\mathrm{p}_\omega = \frac{\mathrm{p}_0}{2\pi v} \cos\left(\frac{\omega_0}{v} z\right) e^{-i\omega z/v} \delta(x)\,\delta(y),$$

$$\mathrm{m}_\omega = \frac{\mathrm{m}_0}{2\pi v} \cos\left(\frac{\omega_0}{v} z\right) e^{-i\omega z/v} \delta(x)\,\delta(y), \qquad (1.1)$$

where $\omega_0 = \sqrt{1-\beta^2}\,\omega_0'$ is the frequency measured in a stationary system of coordinates, and $\mathbf{p}_0$ and $\mathbf{m}_0$ are the electric and magnetic moments of the oscillator in the stationary system of coordinates; they are related to $\mathbf{p}_0'$ and $\mathbf{m}_0'$ by the appropriate relativistic transformation formulas.[1*] The currents and charges at each point of space are determined by the formulas

$$i\omega \mathbf{p}_\omega + c\,\mathrm{curl}\ \mathbf{m}_\omega = \mathbf{j}_\omega, \qquad \mathrm{div}\,\mathbf{p}_\omega = -\rho_\omega \qquad (1.2)$$

Thus the problem reduces to that of finding the electromagnetic field in the medium under consideration for given sources, when their density and therefore the field vectors depend on z through a factor $\exp[-i(\omega \pm \omega_0)z/v]$.

Maxwell's equations for the Fourier components of the field in an anisotropic and gyrotropic medium have the form

$$\mathrm{curl}\,\mathbf{H}_\omega = ik\hat{\varepsilon}\mathbf{E}_\omega + 4\pi \mathbf{j}_\omega / c, \qquad \mathrm{div}\,(\hat{\varepsilon}\mathbf{E}_\omega) = 4\pi\rho_\omega,$$
$$\mathrm{curl}\,\mathbf{E}_\omega = -ik\mathbf{H}_\omega, \qquad \mathrm{div}\,\mathbf{H}_\omega = 0, \qquad (1.3)$$

where

$$\hat{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -ig & 0 \\ ig & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}.$$

The magnetic permeability of the medium is taken equal to unity.

It can be shown that the solution of the system (1.3) can be expressed in terms of two scalar functions ψ_1 and ψ_2, which play the role of potentials for (1.3), and which satisfy the equation

$$\Delta_2 \psi_{1,2} + \sigma_{1,2}^2 \psi_{1,2} = 4\pi f_{1,2}, \qquad (1.4)$$

*We mention that the relativistic electric moment of a polarizable circuit, with a magnetic moment oriented perpendicular to the velocity, is to be calculated by the formula $\mathbf{p}_0 = (\hat{\varepsilon}_1/c)[\mathbf{v} \times \mathbf{m}_0']$, where $\hat{\varepsilon}_1$ is the dielectric tensor of the material of the circuit (as is done in the work of Ginzburg and Éĭdman[8]).

where $\sigma_{1,2}$ are the roots of the equation

$$\sigma^4 - \left[\left(1+\frac{\varepsilon_1}{\varepsilon_2}\right)s^2 - \frac{k^2g^2}{\varepsilon_1}\right]\sigma^2 + \frac{\varepsilon_1}{\varepsilon_2}s^4 - \frac{k^4g^2\varepsilon_2}{\varepsilon_1} = 0, \quad (1.5)$$

and where

$$f_{1,2} = \frac{i\sigma_{1,2}^2}{\varepsilon_2\omega} j_{\omega z} - \frac{\sigma_{1,2}^2 - k^2\varepsilon_2}{\varepsilon_2\mu\omega}\operatorname{div}_2 \mathbf{j}_\omega - \frac{i(\sigma_{1,2}^2 - s^2)\varepsilon_1}{\varepsilon_2\mu\omega g}\operatorname{curl}_z \mathbf{j}_\omega,$$

$$s^2 = \varepsilon_2(k^2\varepsilon_1 - \mu^2)/\varepsilon_1, \qquad \mu = (\omega \pm \omega_0)/v,$$

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \operatorname{div}_2 \mathbf{j}_\omega = \partial j_{\omega x}/\partial x + \partial j_{\omega y}/\partial y. \quad (1.6)$$

The field vectors are found from the relations

$$\mathbf{E}_{\omega t} = \hat{e}_1\nabla\psi_1 - \hat{e}_2\nabla\psi_2 + \frac{4\pi}{c}\hat{p}\mathbf{j}_\omega,$$

$$E_{\omega z} = [(\sigma_2^2 - s^2)\psi_1 - (\sigma_1^2 - s^2)\psi_2]/(\sigma_2^2 - \sigma_1^2),$$

$$\mathbf{H}_\omega = (i/k)\operatorname{curl}\mathbf{E}_\omega, \quad (1.7)$$

where $\mathbf{E}_{\omega t}$ is the transverse component of the field, and where $\hat{e}_{1,2}$ and $\hat{p}$ are defined as operators:

$$\hat{e}_n = \begin{pmatrix} ie_{n1} & e_{n2} \\ -e_{n2} & ie_{n1} \end{pmatrix}, \qquad \hat{p} = \begin{pmatrix} ip_1 & -p_2 \\ p_2 & ip_1 \end{pmatrix},$$

$$e_{n1} = \mu\varepsilon_2(\sigma_n^2 - \varepsilon_1 s^2/\varepsilon_2)/\varepsilon_1\sigma_n^2(\sigma_2^2 - \sigma_1^2),$$

$$e_{n2} = k^2\varepsilon_2\mu g/\sigma_n^2\varepsilon_1(\sigma_2^2 - \sigma_1^2),$$

$$p_1 = ks^2/\sigma_1^2\sigma_2^2, \qquad p_2 = gk^3\varepsilon_2/\varepsilon_1\sigma_1^2\sigma_2^2, \qquad n = 1.2. \quad (1.8)$$

We mention that the proposed method, apart from the case considered here, may prove useful for solution of the problem of excitation of a gyrotropic medium by an arbitrary current.

2. Of greatest interest are two physically distinct orientations of the oscillator: parallel and perpendicular to the z axis. For simplicity we restrict ourselves to these two cases; the solution in the case of arbitrary orientation of the oscillator can be obtained without difficulty from our formulas.

a) Electric oscillator parallel to the z axis. The functions ψ_1 and ψ_2 are found from the equations

$$\nabla_2^2\psi_{1,2} + \sigma_{1,2}^2\psi_{1,2} = -(\sigma_{1,2}^2 p_0/\varepsilon_2 v)\, e^{-i(\omega\pm\omega_0)z/v}\delta(x)\,\delta(y), \quad (2.1)$$

where

$$p_0 = p_0'\sqrt{1-\beta^2}.$$

The solution of (2.1) has the form

$$\psi_{1,2} = (\sigma_{1,2}^2 p_0/4v\varepsilon_2)\, b_{1,2}(r)\, e^{-i(\omega\pm\omega_0)z/v}, \quad (2.2)$$

where

$$b_{1,2}(r) = \begin{cases} -iH_0^{(2)}(\sigma_{1,2}r), & \omega > 0, \\ iH_0^{(1)}(\sigma_{1,2}r), & \omega < 0, \end{cases} \quad (2.3)$$

and where $H_0^{(1)}$ and $H_0^{(2)}$ are the Hankel functions of the first and second kinds. In this case formulas (2.2), (1.7), and (1.8) fully determine the field.

b) Electric oscillator perpendicular to the z axis.

Let the moment of the oscillator be oriented along the x axis. In this case the oscillator acquires a magnetic moment directed along the y axis and equal to $m_y = -\beta p_0'$. On carrying out a calculation similar to the preceding one, we find from (1.4) for this case

$$\psi_{1,2} = \frac{p_0'}{4\omega\varepsilon_2(\omega\pm\omega_0)}\left[i(\sigma_{1,2}^2 \pm k^2\varepsilon_2\omega_0)\cos\varphi \mp \right.$$

$$\left. \mp \frac{\omega_0\varepsilon_1}{g}(\sigma_{1,2}^2 - s^2)\sin\varphi\right]\frac{\partial b_{1,2}}{\partial r}\, e^{-i(\omega\pm\omega)z/v}, \quad (2.4)$$

where φ is a polar angle measured from the x axis. The field vectors can be easily found with the aid of (2.4) and (1.7). Expressions for the components of the field of a magnetic oscillator are found in a completely analogous manner. We remark that the formulas obtained possess great generality. For example, the components of the Cerenkov radiation field of a dipole with moment π_0' can be obtained without difficulty from our formulas by setting $\omega_0 = 0$ and $p_0' = \pi_0'$; the components of the Cerenkov radiation field of an electron, by setting $\omega_0 = 0$ and $p_0 = -ive/\omega$ (see reference 3).

3. In the wave zone, by use of the asymptotic formulas for the Hankel functions, the components of the field can be expressed in the form

$$a\exp\left[-i\left(\frac{\omega\pm\omega_0}{v}z + \sigma_{1,2}r - \omega t\right)\right]. \quad (3.1)$$

If we require that the expressions for the field components have the form of a traveling plane wave with wave normal oriented at angle θ to the axis, i.e., that

$$\omega n\cos\theta/c = (\omega\pm\omega_0)/v, \qquad \omega n\sin\theta/c = \sigma_{1,2}, \quad (3.2)$$

then it is not difficult to obtain the condition for the Doppler frequency,

$$\omega = \omega_0/|1 - \beta n(\omega,\theta)\cos\theta|, \quad (3.3)$$

which agrees with the usual Doppler condition for an isotropic medium; but the refractive index $n(\omega,\theta)$ for the medium under consideration is a complicated function of ω and θ and is different for waves of different polarizations. By substituting (3.2) in (1.5) and solving the biquadratic equation, one can obtain the usual formula for the index of refraction of a gyrotropic medium. Certain results of Frank[1] are also easily generalized to a gyrotropic medium; for instance, the angular width of a spectral line is determined by the formula

$$\Delta\theta = \frac{\pm 2\pi (n_1^2 - n_2^2)(\varepsilon_1 \sin^2\theta + \varepsilon_2 \cos^2\theta)}{T\beta n_{1,2} \sin\theta [\varepsilon_1^2 - g^2 - \varepsilon_1\varepsilon_2 - n_{1,2}^2 (\varepsilon_2 \cos^2\theta + \varepsilon_1 (1 + \sin^2\theta))]}, \tag{3.4}$$

where T is the travel time of the oscillator.

As in an isotropic medium, at oscillator speeds above that of light and at not too large angles, splitting of the Doppler line occurs (the complex Doppler effect). A sufficient condition for this effect is (cf. reference 1)

$$\beta n(\omega, \theta) \cos\theta > 1. \tag{3.5}$$

An estimate of the derivative of the left side of (3.5) shows that the point $\theta = 0$ is a maximum, i.e., a necessary condition for the complex effect at superluminal oscillator speeds can be written in the form

$$(\varepsilon_1 \pm g)\beta^2 > 1. \tag{3.6}$$

For a plasma in a magnetic field,

$$\varepsilon_1 = 1 - \frac{\omega_n^2}{\omega^2 - \omega_H^2}, \quad \varepsilon_2 = 1 - \frac{\omega_n^2}{\omega^2},$$

$$g = \frac{\omega_n^2 \omega_H}{\omega(\omega^2 - \omega_H^2)} \tag{3.7}$$

(where $\omega_n = \sqrt{4\pi Ne^2/m}$ and $\omega_H = eH/mc$ are respectively the plasma and gyromagnetic frequencies), and (3.6) takes the form

$$(1 - \omega_n^2/\omega(\omega \pm \omega_H))\beta^2 > 1. \tag{3.8}$$

It is not difficult to see that (3.8) can hold only when $\omega < \omega_H$ for the extraordinary wave.*

The condition (3.5) is by no means a necessary condition for the complex Doppler effect. In general the effect is possible at speeds much less than the speed of light in the medium, if only the following condition is satisfied (cf. reference 1):

$$\omega_0 \beta [\partial n/\partial\omega]_{\omega=\omega_\theta} \cos\theta > 1. \tag{3.9}$$

As in the case of the superluminal complex effect, a necessary condition for fulfilment of (3.9) is

$$\omega_0 \beta \left[\frac{d}{d\omega} \sqrt{\varepsilon_1 \pm g}\right]_{\omega=\omega_\theta} > 1. \tag{3.1}$$

It has been shown[4] that the complex effect can occu in a plasma in a magnetic field either at frequencii near the gyromagnetic frequency of the plasma, or under the condition $\epsilon_1 \pm g \approx 0$.

4. We consider the energy radiated by a moving oscillator for two different orientations of it. The amount of energy emitted by the oscillator per unit length of its path we determine with the aid of Poynting's theorem, using the asymptotic expressions for the cylinder functions. After somewhat laborious transformations we obtain, for an oscillator oriented along the velocity,

$$\frac{d\mathcal{E}}{dz} = \frac{p_0^2}{4v^2}\left(\int \frac{\sigma_1^2(s^2 - \sigma_2^2)}{\varepsilon_2(\sigma_2^2 - \sigma_1^2)}\,\omega\,d\omega + \int \frac{\sigma_2^2(s^2 - \sigma_1^2)}{\varepsilon_2(\sigma_1^2 - \sigma_2^2)}\,\omega\,d\omega\right), \tag{4.1}$$

where the first and second integrals extend over the frequency ranges $\sigma_1^2(\pm\omega_0) > 0$ and $\sigma_2^2(\pm\omega_0) > 0$, respectively, and $\omega > 0$. The sign in $\pm\omega_0$ means that the parentheses in (4.1) actually contain a sum of four terms; $+\omega_0$ occurs in the first two and $-\omega_0$ in the second two. We remark that the terms with $+\omega_0$ differ from zero only at superluminal oscillator speeds.

A similar calculation gives the amount of energy emitted by an oscillator with moment oriented perpendicular to the velocity; in this case it is possible to obtain the distribution of energy density with angle φ. After a series of laborious calculations we get

$$\frac{d\mathcal{E}(\varphi)}{dz} = \frac{p_0'^2}{8\pi v^2}\left[\int\limits_{\sigma_1^2(\pm\omega_0)>0} \frac{[g^2(\sigma_1^2\omega \pm k^2\varepsilon_2\omega_0)^2 \cos^2\varphi - \varepsilon_1^2\omega_0^2(\sigma_1^2 - s^2)^2 \sin^2\varphi](s^2 - \sigma_2^2)}{g^2\varepsilon_2\omega(\omega \pm \omega_0)^2(\sigma_2^2 - \sigma_1^2)}\,d\omega\right.$$

$$\left. + \int\limits_{\sigma_2^2(\pm\omega_0)>0} \frac{[g^2(\sigma_2^2\omega \pm k^2\varepsilon_2\omega_0)^2 \cos^2\varphi - \varepsilon_1^2\omega_0^2(\sigma_2^2 - s^2)^2 \sin^2\varphi](s^2 - \sigma_1^2)}{g^2\varepsilon_2\omega(\omega \pm \omega_0)^2(\sigma_1^2 - \sigma_2^2)}\,d\omega\right]. \tag{4.2}$$

The total energy can be obtained without difficulty by integration over φ.

We note the great generality of the formulas obtained. In fact, by setting $\omega_0 = 0$ and $p_0^2/2 = \pi_0^2$ we obtain a formula for the energy of the Cerenkov radiation of a constant dipole with moment π_0 in the proper coordinate system:

$$\frac{d\mathcal{E}}{dz} = \frac{\pi_0'^2(1 - \beta^2)}{v^4}\left[\int\limits_{\varkappa_1^2>0} \frac{\varkappa_1^2(\varepsilon_2(\varepsilon_1\beta^2 - 1) - \varepsilon_1\varkappa_2^2)}{\varepsilon_1\varepsilon_2(\varkappa_2^2 - \varkappa_1^2)}\,\omega^3\,d\omega\right.$$

$$\left. + \int\limits_{\varkappa_2^2>0} \frac{\varkappa_2^2(\varepsilon_2(\varepsilon_1\beta^2 - 1) - \varepsilon_1\varkappa_1^2)}{\varepsilon_1\varepsilon_2(\varkappa_1^2 - \varkappa_2^2)}\,\omega^3\,d\omega\right],$$

$$\varkappa_{1,2}^2 = \frac{\omega^2}{v^2}\varkappa_{1,2}^2 = \frac{\omega^2}{2v^2}\left((\varepsilon_1\beta^2 - 1)\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right) - \frac{g^2\beta^2}{\varepsilon_1}\right.$$

$$\left. \pm \left[\left[(\varepsilon_1\beta^2 - 1)\left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right) + \frac{g^2\beta^2}{\varepsilon_1}\right]^2 + 4\frac{g^2\beta^2}{\varepsilon_1}\right]^{1/2}\right) \tag{4.3}$$

*By extraordinary and ordinary waves will be understood those waves that reduce for g = 0 to the corresponding waves in a uniaxial crystal.

for a dipole oriented parallel to the velocity, and

$$\frac{d\mathcal{E}(\varphi)}{dz} = \frac{\pi_0'^2 \cos^2\varphi}{2\pi v^4}\left[\int\limits_{x_1^2>0} \frac{x_1^4(\varepsilon_2(\varepsilon_1\beta^2-1)-\varepsilon_1 x_2^2)}{\varepsilon_1\varepsilon_2(x_2^2-x_1^2)}\omega^3\,d\omega + \int\limits_{x_2^2>0} \frac{x_2^4(\varepsilon_2(\varepsilon_1\beta^2-1)-\varepsilon_1 x_1^2)}{\varepsilon_1\varepsilon_2(x_1^2-x_2^2)}\omega^3\,d\omega\right] \quad (4.4)$$

for a dipole oriented perpendicular to the velocity. And finally, by setting $\omega_0 = 0$ and $p_0' = -\text{ive}/\omega$ and substituting in (4.1), a formula is automatically obtained for the energy of a Cerenkov electron flying at superluminal speed along the z axis (cf. reference 3).

5. By way of illustration, we consider a uniaxial crystal. On setting $g = 0$ in (4.1) and (4.2), we get

$$\frac{d\mathcal{E}_\parallel}{dz} = \frac{p_0^2}{4c^2v^2}\int\limits_{s^2(\pm\omega_0)>0}\left(\omega^2 - \frac{(\omega\pm\omega_0)^2}{\varepsilon_1\beta^2}\right)\omega\,d\omega \quad (5.1)$$

for an oscillator parallel to the velocity, and

$$\frac{d\mathcal{E}_\perp}{dz} = \frac{p_0'^2}{8c^4}\left[\int\limits_{s^2(\pm\omega_0)>0}\varepsilon_2\left(\omega - \frac{\omega\pm\omega_0}{\varepsilon_1\beta^2}\right)\omega\,d\omega + \frac{\omega_0}{\beta^2}\int\limits_{(\varepsilon_1/\varepsilon_2)s^2(\pm\omega_0)>0}\omega\,d\omega\right] \quad (5.2)$$

for an oscillator perpendicular to the velocity.

We notice that for $\omega_0 = 0$, $\epsilon_1 = \epsilon_2$, and $p_0^2/2 = \pi_0^2$, formulas for the energy of Cerenkov radiation are obtained that agree with those obtained by Frank.[5]

If we neglect dispersion — that is, if the constants of the medium change little in the range of the Doppler frequencies emitted by the source — then it is possible to carry out the integration in (5.1) and (5.2). After simple calculations, (5.1) and (5.2) take the form, for $\beta\sqrt{\epsilon_1} < 1$,

$$d\mathcal{E}_\parallel/dz = \omega_0'^4 p_0'^2\sqrt{\varepsilon_1}(1-\beta^2)^3/3c^3v(1-\beta^2\varepsilon_1)^3, \quad (5.3)$$

$$d\mathcal{E}_\perp/dz = \omega_0'^4 p_0'^2(\varepsilon_2+3\varepsilon_1)/12\,c^3v\sqrt{\varepsilon_1}(1-\beta^2\varepsilon_1)^2. \quad (5.4)$$

For $\beta\sqrt{\epsilon_1} > 1$, the integrals in (5.1) and (5.2) diverge; this is connected with the fact that in this case it is not permissible to neglect dispersion. We note that (5.3) indicates that the total radiation energy is independent of ϵ_2. Comparison of (5.3) and (5.4) also shows an essential dependence of the radiated energy on the orientation of the oscillator. For $\epsilon_1 = \epsilon_2$, (5.3) and (5.4) reduce to the corresponding formulas for the radiation energy of an oscillator in an isotropic medium (cf. reference 1).

In a medium without dispersion, the frequency radiated by the oscillator is simply related to the angle at which this frequency is observed. That is, if in (5.1) and (5.2) we transform from integration over ω to integration over angles with the aid of

$$\omega_{1,2} = \omega_0/|1-\beta n_{1,2}(\theta)\cos\theta|, \quad (5.5)$$

where

$$n_1(\theta) = \sqrt{\varepsilon_1}; \qquad 1/n_2^2(\theta) = \cos^2\theta/\varepsilon_1 + \sin^2\theta/\varepsilon_2,$$

and if we consider the path of the oscillator to be finite and of length l (cf. reference 2), then we can obtain a formula for the distribution of the radiation energy with angle θ. After simple transformations it follows from (5.1) and (5.2) that

$$W_\parallel(\theta) = p_0^2 l\omega_0^4 n_2^5(\theta)\sin^2\theta/8c^3v\varepsilon_2^2|1-n_2(\theta)\beta\cos\theta|^5 \quad (5.6)$$

for an oscillator oriented along the velocity, and

$$W_\perp(\theta,\varphi) = \frac{p_0'^2\omega_0^4 l}{8c^3v}\left[\frac{(\varepsilon_1\beta - n_2(\theta)\cos\theta)^2 n_2^3(\theta)\cos^2\varphi}{\varepsilon_1^2|1-\beta n_2(\theta)\cos\theta|^5} + \frac{\sqrt{\varepsilon_1}\sin^2\varphi}{|1-\beta\sqrt{\varepsilon_1}\cos\theta|^3}\right] \quad (5.7)$$

for an oscillator oriented perpendicular to the velocity. For $\epsilon_1 = \epsilon_2$, (5.6) and (5.7) reduce to the analogous formulas for an isotropic medium (cf. reference 1).

6. The Doppler effect in a uniaxial gyrotropic crystal has a number of peculiarities as compared with an isotropic medium. As was mentioned by Pafomov,[6] in the range of frequencies in which the radial group velocity is negative (i.e., the projections of the group velocity and of the wave vector along the radius have opposite signs), in order that the solution shall have physical meaning it is necessary to use advanced potentials. If in this case $\cos\theta$ and $\cos\varphi$ (φ is the angle between the z axis and the direction of the beam) have the same sign, there arise a peculiar "inverted" Doppler effect — that is, lower frequencies are radiated forward and higher frequencies backward. This same effect occurs if $W_r > 0$ and if $\cos\theta$ and $\cos\varphi$ have opposite signs.

The projection W_r of the group velocity along the radius, in a gyrotropic medium, has the form[7]

$$W_r = -\frac{\partial n\cos\theta/\partial\theta}{(n/c)\,\partial\omega n/\partial\omega} = \frac{c\,|n^2(\varepsilon_2\cos^2\theta + \varepsilon_1(1+\sin^2\theta)) - \varepsilon_1^2 + g^2 - \varepsilon_1\varepsilon_2]\sin\theta}{\frac{\partial}{\partial\omega}(\omega n)\,[2n^2(\varepsilon_1\sin^2\theta + \varepsilon_2\cos^2\theta) - (\varepsilon_1 - g^2)\sin^2\theta - \varepsilon_1\varepsilon_2(1+\cos^2\theta)]}, \quad (6.2)$$

and the angles θ and φ are connected by the relation

$$\tan\varphi = -\frac{\partial n\cos\theta/\partial\theta}{\partial n\sin\theta/\partial\theta}$$

$$= \frac{[n^2(\varepsilon_2\cos^2\theta+\varepsilon_1(1+\sin^2\theta))-\varepsilon_1^2+g^2-\varepsilon_1\varepsilon_2]\tan\theta}{n^2(\varepsilon_1\sin^2\theta+\varepsilon_2\cos^2\theta+\varepsilon_2)-2\varepsilon_1\varepsilon_2}. \quad (6.3)$$

As was mentioned in reference 4, in a plasma with a magnetic field at angles θ close to $\pi/2$, in the range of frequencies

$$[1/2(\omega_H^2+\sqrt{4\omega_n^4+\omega_H^4})]^{1/2} < \omega < \sqrt{\omega_n^2+\omega_H^2} \quad (6.4)$$

an "inverted" Doppler effect occurs. For a more general conclusion a more detailed study of formulas (6.2) and (6.3) is required.

We point out, furthermore, that in a uniaxial anisotropic crystal without optical activity, if the crystal constants are determined from an oscillator model (as was done in reference 6), there exists a frequency range in which an "inverted" Doppler effect can occur at any angles.

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ONE-PARTICLE MECHANISM IN PHOTONUCLEAR REACTIONS

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The one-particle (direct) mechanism of photonuclear reactions at high energies is investigated on the basis of the shell model. It is shown that the ground state momentum distribution obtained from that model allows one to explain the forward shift of the maximum in the angular distribution of the photoprotons and leads to a correct magnitude of the reaction cross section.

1. INTRODUCTION

THE study of photonuclear reactions at large energies ($\gtrsim 20$ Mev) of the emitted nucleons is important because it gives a direct method of investigating the correlations of the nucleons and their momentum distribution in the nuclear ground state. Of equal interest is the question on the interaction of the electromagnetic field with the nucleus at high photon and reaction product energies. The characteristic peculiarity of photonuclear reactions lies in the fact that the photon carries a momentum which is several times smaller than the momentum of the emitted particle. One therefore has to provide for such a reaction mechanism which will secure the fulfilment of the conservation rules. The two nucleon ("quasi-deuteron") model which was proposed by Levinger[1] and further developed by Dedrik[2] and Gottfried[3] has been since fully confirmed experimentally[4,5] and is generally accepted as the model of photonuclear reactions at high energies. However, there exist also experimental data not contained within the framework of the two nucleon mechanism. These are: (i) the presence of photoprotons with energy almost equal to the maximum energy of the x-ray spectrum;[5,6] (ii) forward shift of the peak in the angular distribution into the range $20 - 50°$;[6,7] (iii) direct observations of the (γ, p) and (γ, n) reactions at high photon energies.[8]

In connection with the indicated facts the importance of the (one-nucleon) direct photoeffect has been repeatedly mentioned. In particular, this idea has been successfully applied to the region of the giant resonance.[9]

In the present paper the one-nucleon direct photoeffect will be investigated on the basis of the shell model in its simplest form. The conservation of momentum will be taken care of by considering that a bound nucleon possesses momentum (the internal momentum distribution). Furthermore the forward shift of the angular distribution will find an explanation analogous to the case of the atomic photoeffect.[10] The well known success of the shell model justifies the hope that its wave functions will sufficiently accurately give the ground state momentum distribution of the nucleons.

2. THE REACTION MECHANISM AND THE CROSS SECTIONS

The basic assumptions which we will make are the following:

1. The nucleus in its ground state can be described as a system of nucleons moving independently in a certain spherically symmetrical potential. The state of each nucleon is described by the orbital angular momentum l, its projection m, and the energy ϵ_l. The spin and magnetic moment of the nucleon will not be taken into account. This in the present case is equivalent to the neglect of the spin-orbit interaction. It should be mentioned that the disregard of the nucleon-nucleon interactions allows only to consider excited "hole" states of the daughter nucleus; the excitation energy of these states is included into the binding energy of the particular nucleon.

2. The final state interaction is given by the optical model of the nucleus.

3. The impulse approximation is applicable.

In a system of noninteracting particles only transitions involving the excitation of a single particle can be induced by a one-particle operator like the interaction operator of the electromagnetic field with a system of charged particles. If we separate mentally the particle that makes the

transition into an excited state, then we can say that the remainder of the nucleus acts rigidly with respect to the electromagnetic disturbance, i.e., the electromagnetic field acts coherently on the nucleons that do not participate in the transition. It is further known that the demand of keeping the center of mass of the nucleus fixed leads to the appearance of the so-called "effective charge" for both protons and neutrons. This can be understood as due to the absorption of photons by the recoiling remainder of those nucleons which do not make a transition. Thus there are two possible photon absorption processes by nuclei within the framework of the one-particle mechanism: absorption of a photon by the nucleon making the transition and absorption of a photon by the remainder of the nucleons not participating in the transition. Clearly, a process of the first kind can occur only for the case of proton emission. The above can be expressed as the following assumption:

4. The interaction operator of the electromagnetic field with the nucleus can be written in the form

$$V = \mathbf{j}\cdot\left[\frac{e_1}{m_1}\, e^{i\mathbf{k}_\omega\cdot\mathbf{r}_1}\,\mathbf{p}_1 + \frac{Ze-e_1}{M-m_1}\, e^{i\mathbf{k}_\omega\cdot\mathbf{r}_2}\mathbf{p}_2\right], \qquad (1)$$

where e_1, m_1 and $\mathbf{r}_1$ — the charge, mass and radius vector respectively of the nucleon making the transition, Ze and M are the charge and mass of the nucleus, $\mathbf{r}_2$ is the radius vector of the center of mass of the daughter nucleus, and $\mathbf{j}$ is the polarization vector of the photon. The necessary normalization of the electromagnetic field can be achieved by adding the factor $(2\pi\hbar/c\omega)^{1/2}$ in (1).

We now introduce the center-of-mass system of the photon and the nucleus. In this system the nucleus moves with momentum $\mathbf{K} = -\mathbf{k}_\omega$ and the two parts of the nucleus move with respect to its center of mass with momenta χ_0 and $-\chi_0$. Then the wave function of the initial state can be written as

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = (2\pi)^{-3/2}\psi(\mathbf{r})\, e^{i\mathbf{K}\cdot\mathbf{R}}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,$$
$$\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M.$$

After absorption of the photon, the wave function describing the motion of the photonucleon and the daughter nucleus is $\Phi(\mathbf{r}_1, \mathbf{r}_2) = (2\pi)^{-3/2}\varphi(\mathbf{r})$.

As is well known, the reaction cross section is given by

$$d\sigma = \frac{2\pi}{\hbar}|M|^2\rho(E), \qquad (2)$$

$$M = \int \Phi^+(\mathbf{r}_1\,\mathbf{r}_2)\, V\Psi(\mathbf{r}_1, \mathbf{r}_2)\, d\mathbf{r}_1 d\mathbf{r}_2, \quad \rho(E) = k^2\frac{dk}{dE}d\Omega, \qquad (2')$$

where $\mathbf{k}$ is the relative momentum in the final state.

We introduce

$$M' = (2\pi)^3\left[\frac{e_1}{m_1}\int \Phi^+(\mathbf{r}_1, \mathbf{r}_2)\, e^{i\mathbf{k}_\omega\cdot\mathbf{r}_1}\mathbf{j}\cdot\nabla_1\Psi(\mathbf{r}_1, \mathbf{r}_2)\, d\mathbf{r}_1 d\mathbf{r}_2 + \frac{Ze-e_1}{M-m_1}\int \Phi^+(\mathbf{r}_1, \mathbf{r}_2)\, e^{i\mathbf{k}_\omega\cdot\mathbf{r}_2}\mathbf{j}\cdot\nabla_2\Psi(\mathbf{r}_1, \mathbf{r}_2)\, d\mathbf{r}_1 d\mathbf{r}_2\right] \qquad (3)$$

and go over to momentum space.

We write

$$\Phi(\mathbf{r}_1, \mathbf{r}_2) = (2\pi)^{-3}\int C(\mathbf{k}_1, \mathbf{k}_2)\, e^{i(\mathbf{k}_1\cdot\mathbf{r}_1 + \mathbf{k}_2\cdot\mathbf{r}_2)}d\mathbf{k}_1 d\mathbf{k}_2$$

and insert this into (3). Going over to relative coordinates and recalling that the photon polarization is transverse and $\mathbf{K} = -\mathbf{k}_\omega$, we obtain

$$M' = \int d\mathbf{k}_1 C(\mathbf{k}_1, -\mathbf{k}_1)\left[\frac{e_1}{m_1}\int e^{-i(\mathbf{k}-\mathbf{k}''_\omega)\cdot\mathbf{r}}\,\mathbf{j}\cdot\nabla\psi(\mathbf{r})\, d\mathbf{r} - \frac{Ze-e_1}{m_2}\int e^{-i(\mathbf{k}+\mathbf{k}'_\omega)\cdot\mathbf{r}}\,\mathbf{j}\cdot\nabla\psi(\mathbf{r})\, d\mathbf{r}\right], \qquad (4)$$

where

$$\mathbf{k}'_\omega = m_1\mathbf{k}_\omega/M, \quad \mathbf{k}''_\omega = m_2\mathbf{k}_\omega/M,$$
$$\mathbf{k} = (m_2\mathbf{k}_1 - m_1\mathbf{k}_2)/M \qquad (m_2 = M - m_1).$$

Clearly $C(\mathbf{k}_1, -\mathbf{k}_1)$ is the amplitude of the momentum distribution of the nucleon which is making the transition. It can be written in the form

$$C(\mathbf{k}_1, -\mathbf{k}_1) \equiv C(\mathbf{k}_1) = (2\pi)^{-3/2}\int e^{-i\mathbf{k}_1\cdot\mathbf{r}}\varphi(\mathbf{r})\, d\mathbf{r}. \qquad (5)$$

Further, in the spirit of the impulse approximation and taking into account the kinematics of both kinds of processes, we put in (4) $\mathbf{k} = \mathbf{k}_{10}$, where $\mathbf{k}_{10}$ is the momentum of the emitted nucleon right after absorbing the photon: then both $\mathbf{k}_{10} - \mathbf{k}''_\omega$ and $\mathbf{k}_{10} + \mathbf{k}'_\omega$ represent that initial (internal) momentum which is necessary for the fulfilment of the momentum conservation law. Integrating (4) by parts we obtain the matrix element M' in the form

$$M' = (2\pi)^{3/2}\int d\mathbf{k}_1 C^+(\mathbf{k}_1)\left[\frac{e_1}{m_1}G(\mathbf{k}_{10} - \mathbf{k}''_\omega) - \frac{Ze-e_1}{m_2}G(\mathbf{k}_{10} + \mathbf{k}'_\omega)\right]\mathbf{j}\cdot\mathbf{k}_{10}, \qquad (6)$$

where

$$G(\mathbf{p}) = (2\pi)^{-3/2}\int e^{-i\mathbf{p}\cdot\mathbf{r}}\psi(\mathbf{r})\, d\mathbf{r} \qquad (7)$$

is the amplitude of the momentum distribution in the nuclear ground state.

3. COMPARISON WITH EXPERIMENT

For the comparison with experiment one has to make a choice of initial and final state. As is well known for light nuclei the oscillator potential gives good results when applied to the ground state. The solution of the Schrödinger equation in momentum space gives for the momentum distribution

of a state with principal quantum number unity the following expression:

$$G(p_0) = C \exp(-p_0^2/2\mu\hbar\omega_0)\, p_0^l Y_{lm}(p_0/p_0), \qquad (8)$$

where

$$C = \hbar^{3/2}\sqrt{2/\Gamma(l+3/2)}\,(1/\mu\hbar\omega_0)^{(2l+3)/4},$$

$$p_0'' = \hbar^2\chi_0^2 = \hbar^2(k_{10}^2 + k_\omega''^2 - 2k_{10}k_\omega'' \cos\theta)$$

(absorption of type 1)

$$p_0^2 = \hbar^2\chi_0^2 = \hbar^2(k_{10}^2 + k_\omega'^2 + 2k_{10}k_\omega' \cos\theta)$$

(absorption of type 2)

θ is the angle between $\mathbf{k}_\omega$ and $\mathbf{k}_{10}$, and μ is the reduced mass. The only parameter in (8) — the characteristic oscillator frequency ω_0 — is determined by the condition that the rms radius of the nucleus agree with the experiment.[11]

The determination of $C(\mathbf{k}_1)$ turns out to be a difficult problem. In the considered energy region the known approximate methods are not applicable; the exact computation of $C(\mathbf{k}_1)$ requires the solution of an integral equation of complicated character. It should however be borne in mind that the choice of the form and the parameters of the interaction in the final state at large energies is rather problematical. Therefore an exact complicated calculation would actually not have a greater value than an approximate estimate based on simple concepts. This estimate is easy to perform with the assumption that the final state interaction can be described by an optical model. This provides in essence two effects: (1) The imaginary part of the potential leads to a decrease of the cross section of the direct photoeffect because of the appearance of cascades within the nucleus and (or) the excitation of compound nucleus type excited states; the latter receives also a contribution from the reflection from the barrier. (2) After absorption of the photon, the nucleon experiences elastic scattering due to the real part of the potential, which will lead to a certain smearing out of the angular distribution.

The first effect can be easily estimated from the mean free path of the nucleon within the nucleus, the second — from the experimental elastic scattering cross sections. At high energies the barrier has a minor influence. This way one can as a rough approximation assume a plane wave for the final state wave function: $\varphi(\mathbf{r}) = \exp(i\mathbf{k}\cdot\mathbf{r})$; then $C(\mathbf{k}_1) = (2\pi)^{3/2}\delta(\mathbf{k}-\mathbf{k}_1)$. In order to determine the momentum of the photonucleon within the nucleus, $\mathbf{k}_{10}$, one still needs to know the depth of the potential well. Taking for simplicity a square well we have $k_{10}^2 = 2m\hbar^{-2}(E_p + V - \epsilon_l)$ where V — the depth of the well; its value can be obtained from the analysis of the elastic proton (neutron) scattering in the framework of the optical model.[12]

Assuming circular polarization and averaging over the initial states, we obtain for the cross section of the one-particle photoreaction on a particle with orbital angular momentum l, in the center-of-mass system (for the (γ, p) reaction)

$$d\sigma_l = \frac{e^2}{\hbar c}\,\frac{\pi^{1/2}}{(\mu\hbar\omega_0)^{5/2}}\,\frac{\hbar^2 p_{10}^2 p_1}{mE_\omega}$$

$$\times \sin^2\theta\,|A^l(1) - A^l(2)|^2(1-F)R_l\,d\Omega, \qquad (9)$$

where $p_1 = \sqrt{2mE_p}$, E_p is the energy of the photonucleon in the center of mass system, E_ω is the photon energy;

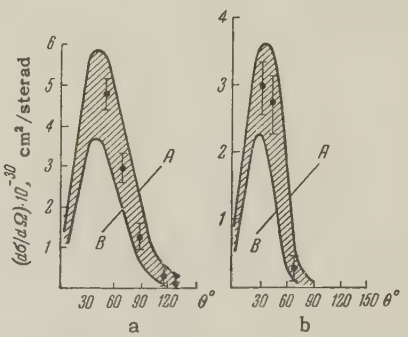
$$A^l(1) = \exp(-p_0^2/2\mu\hbar\omega_0)\, p_0^l(\mu\hbar\omega_0)^{-l/2},$$

$$p_0^2 = p_{10}^2 + p_\omega''^2 - 2p_{10}p_\omega'' \cos\theta;$$

$$A^l(2) = \frac{Z-1}{A-1}\exp(-p_0^2/2\mu\hbar\omega_0)p_0^l(\mu\hbar\omega_0)^{-l/2},$$

$$p_0^2 = p_{10}^2 + p_\omega'^2 + 2p_{10}p_\omega' \cos\theta.$$

F is a coefficient describing the absorption of the nucleons, $R_0 = 1$, $R_1 = 2/3$, and $R_2 = 4/15$. For (γ, n) reactions $A^l(1) \equiv 0$, and the factor $(Z-1)/(A-1)$ has to be replaced by $Z/(A-1)$.



Points: experimental data of Whitehead et al.,[6] curve A: calculated with a potential $V = V_{RW} - 5$ Mev, B: calculated with a potential $V = V_{RW} + 5$ Mev, where V_{RW} is the potential (set B) of Riesenfeld and Watson.[13] a: $E_p = 37$ Mev, $E_\omega = 45$ to 56 Mev; b: $E_p = 78$ Mev; $E_\omega = 90$ to 110 Mev.

We shall compare our results with the experiment for the C^{12} nucleus. We further choose those data where the (γ, pn) reaction cannot take place because of energetic reasons. From the figure one can see that one can obtain agreement with the experimental points by varying the depth of the potential well for the final state.

It should be mentioned that according to calculation the transitions from the $1s_{1/2}$ orbit in C^{12} do not contribute more than 10% to the total cross section. Therefore the assumption that the nu-

cleons do not interact, which is particularly bad for nucleons in the $1s_{1/2}$ state, cannot essentially influence the results.

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ON THE SHAPE OF EVEN-EVEN NUCLEI

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A nuclear model assuming a core plus two nucleons in a shell with angular momentum j is considered. The energy is determined as a function of the parameters β and γ for various values of the total angular momentum of the nucleons. It is shown that the minimum energy in the ground state corresponds to a shape of the nucleus without axial symmetry, provided that $j > 3/2$.

INTRODUCTION

FOR a long time physicists were convinced that the atomic nuclei have spherical shape. The success of the liquid drop model of the nucleus strengthened this belief. However, the detailed study of the ground and excited states of the nucleus established the fact that many nuclei deviate from the spherical shape. In particular, this holds for nuclei whose mass numbers lie in the regions $A > 225$, $155 < A < 185$, and $A \sim 24$.

The nonspherical shape of the nucleus manifested itself in the presence of the rotational spectrum for the excited states, and in the large electric quadrupole moments of the stationary states of the nucleus, in the measurement of which great progress has been made thanks to the method of nuclear Coulomb excitation and the investigation of the γ transitions in nuclei.

The sizable deviation of the equilibrium shape of the nucleus from spherical symmetry remained unexplained for a long time. The first interpretation was given by J. Rainwater[1] based on a study of the interaction of the nuclear surface with the outer nucleons, i.e., the nucleons which do not belong to completely filled shells. However, it is assumed in this and many later papers[2-5] that the nucleus preserves its axial symmetry. Formally, this amounts to neglecting those parts of the Hamiltonian which are not diagonal in the quantum numbers of the projection of the angular momentum on one of the nuclear axes. The energy of the interaction of the outer nucleons with the nuclear surface therefore was averaged in effect only over nucleon states with a definite value for the angular momentum projection on this axis.

It was shown recently[6] that many properties of the first excited states of even-even nuclei (the order of succession of the spins of the excited states, their energies, and the probabilities for electromagnetic transitions between them) can be readily explained by assuming that the equilibrium shape of the nucleus can in first approximation be represented by a three-axial ellipsoid. The nuclear ellipsoid of Bohr is characterized by the two parameters β and γ; the relations

$$a_0 = \beta \cos\gamma, \qquad a_1 = a_{-1} = 0, \qquad a_2 = a_{-2} = (\beta/\sqrt{2}) \sin\gamma$$

connect these parameters with the parameters a_μ defining the shape of the nucleus:

$$R(\vartheta, \varphi) = R_0 + R_0 \sum_{\mu=-2}^{2} a_\mu Y_{2\mu}(\vartheta, \varphi),$$

in the coordinate system attached to the nucleus. Varying the "asymmetry" parameter γ from 0 to $\pi/3$, with a fixed value β, induces a change of the nuclear shape from a prolate to an oblate elliposid of revolution. The value $\gamma = 30°$ corresponds to a shape which is intermediate between the prolate and the oblate ellipsoids of revolution. In order to obtain agreement with experiment, it had to be assumed in reference 6 that in some nuclei the equilibrium value of γ can reach values close to 30°. This large deviation from axial symmetry calls for a theoretical justification.

The first indications of the possibility that the equilibrium shape of the nuclei may deviate from axial symmetry came from the calculations of Gursky,[7] the results of which were quoted in the paper of Wilets and Jean.[8] These calculations showed that the minimal energy of a nucleon system consisting of 55 protons and 91 neutrons moving in the field of a three-axial ellipsoid corresponds to the values $\beta = 0.04$ and $\gamma = 7.5$. Similar calculations in the same approximation were carried out by Geĭlikman[9] for a three-dimensional

oscillator potential and by Zaikin[10] for a square-well potential. In these papers it is also shown that the minimum energy of the nucleon system corresponds, in a number of cases, to a nuclear shape without axial symmetry. Unfortunately, these numerical estimates lose in value owing to the fact that the spin-orbit interaction is neglected and a special form of the potential is chosen.

In the present paper we propose a new method for the explanation of possible deviations of the equilibrium shape of the nucleus from axial symmetry, which is based on a generalization (to nuclei without axial symmetry) of the method of Bohr.[2]

1. FORMULATION OF THE PROBLEM

We consider a system consisting of a certain number of nucleons forming the core of the nucleus plus two equivalent outer nucleons in a shell with the definite angular momentum j. According to the Pauli principle the total angular momentum of the two nucleons can take only even values:

$$J = 0,\ 2,\ldots\ 2j-1. \tag{1.1}$$

If, in the zeroth approximation, the coupling with the nuclear surface deformation is neglected (the nucleons move in the field of the nuclear core), the total angular momentum of the nucleon J and its projection M on the ζ axis of the coordinate system $\xi\eta\zeta$ attached to the nucleus are integrals of the motion, whose energy is determined by the operator H_p. States ψ_{JM} differing in the values of J and M belong to the same energy, i.e.,

$$(H_p - E_p)\psi_{JM} = 0. \tag{1.2}$$

In accordance with the unified model of Bohr and Mottelson we assume further that the properties of the nuclear core are determined by the operators of the collective motion

$$H_r = \frac{1}{2}\left\{a\,(I_\xi - J_\xi)^2 + b\,(I_\eta - J_\eta)^2 + c\,(I_\zeta - J_\zeta)^2\right\}, \tag{1.3}$$

$$H_{vib} = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right) + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\left(\sin 3\gamma\frac{\partial}{\partial\gamma}\right)\right\} + \frac{C}{2}\beta^2, \tag{1.4}$$

where

$$a = \hbar^2\,[4B\beta^2\sin^2(\gamma - 2\pi/3)]^{-1},$$

$$b = \hbar^2\,[4B\beta^2\sin^2(\gamma + 2\pi/3)]^{-1},\quad c = \hbar^2\,[4B\beta^2\sin^2\gamma]^{-1}, \tag{1.5}$$

C is the elastic constant of the nuclear surface, B is the mass parameter, I_ξ, I_η, I_ζ are the projections of the total angular momentum of the nucleus, and J_ξ, J_η, J_ζ are the projections of the total angular momentum of the outer nucleons.

We assume that the pair of outer nucleons interacts with the core as a whole. Then the operator corresponding to this interaction can be written in the form

$$H_{int} = T\beta\,[\cos\gamma\,(3J^2_\zeta - J^2) + \sqrt{3}\sin\gamma\,(J^2_\xi - J^2_\eta)], \tag{1.6}$$

where T is a parameter which determines the strength of the coupling between the nucleon pair in the j shell and the nuclear surface.

The assumptions at the basis of the expression (1.6) require, of course, a detailed justification. They represent a different limiting case from that considered in the paper of Ford,[5] where it is assumed that the outer nucleons interact independently with the nuclear surface. The H_{int} introduced by Ford completely ignores the interaction between the outer nucleons. Ford's Hamiltonian can therefore not be used in the study of the effects connected with the pairing of the nucleons. In the present paper we shall postulate the interaction (1.6). Moreover, we shall not neglect (as was done in references 2 to 5) the terms in the Hamiltonian which give rise to nondiagonal matrix elements with respect to the magnetic quantum numbers, i.e., we shall not assume that $j_3 = \Omega$ is a good quantum number. Therefore we shall not restrict the class of admissible nucleon states to only those states for which

$$\sum_i \Omega_i = 0. \tag{1.7}$$

In those papers in which only states satisfying the subsidiary condition (1.7) are considered, the equilibrium shape of the nucleus will, of course, always turn out to be axially symmetric.

If the motion connected with the changes in β and γ is slow in comparison with the motion of the outer nucleons, we can apply the adiabatic approximation (which was also used in references 2 to 5). Thus we calculate the energy of the whole system for fixed but arbitrary values β and γ, and determine the values β_0 and γ_0 for which the energy becomes a minimum. These values, then, will also determine the equilibrium shape of the nucleus.

In the adiabatic approximation we can neglect in (1.4) the operators corresponding to the kinetic energy. Then the operator for the surface oscillations takes the form

$$H^a_{vib} = {}^1/_2\,C\beta^2. \tag{1.4a}$$

The total angular momentum of an even-even nucleus in the ground state is equal to zero. The operator for the rotation energy (1.3) in the ground state can therefore be written in the form

$$H_r^0 = {}^1/_2\{aJ_\xi^2 + bJ_\eta^2 + cJ_\zeta^2\}. \qquad (1.3a)$$

The total Hamiltonian for the ground state of the nucleus in the adiabatic approximation then has the form

$$H = H_p + H_r^0 + H_{vib}^a + H_{int}. \qquad (1.8)$$

In the next section we shall find the solution to the equation

$$[H - E(\beta, \gamma)]\psi = 0, \qquad (1.9)$$

which determines the energy of the system as a function of β and γ.

2. CALCULATION OF THE NUCLEAR ENERGY AS A FUNCTION OF β AND γ

For the determination of the energy $E(\beta, \gamma)$ in Eq. (1.9) we write the wave function ψ in the form

$$\psi = \sum_{JM} a_{JM} |JM\rangle,$$

$$|JM\rangle = [(1+\delta_{M0})2]^{-1/2}(\psi_{JM} + \psi_{J,-M}), \qquad (2.1)$$

where J runs through the values (1.1), and M = 0, 2, 4, . . . , J. Substituting (2.1) in (1.9), we obtain a system of equations for the determination of the coefficients a_{JM}. The secular equation for this system determines $E(\beta, \gamma)$.

The non-zero matrix elements of the operators (1.3a) and (1.6) are

$$\langle JM|H_r^0|JM\rangle = \frac{a+b}{4}\{J(J+1) - M^2\} + \frac{c}{2}M^2,$$

$$\langle J, M+2|H_r^0|JM\rangle = \langle JM|H_r^0|JM+2\rangle = \frac{a-b}{8}F(JM),$$

$$\langle JM|H_{int}|JM\rangle = T\beta\cos\gamma\,[3M^2 - J(J+1)],$$

$$\langle JM|H_{int}|JM+2\rangle = \frac{\sqrt{3}}{2}T\beta F(JM)\sin\gamma, \qquad (2.2)$$

where

$$F(JM) = \{(1+\delta_{M0})(J-M)$$
$$\times(J-M-1)(J+M+1)(J+M+2)\}^{1/2},$$

From this we see that J is an integral of the motion, so that the equation for $E(\beta, \gamma)$ splits up into a number of simpler equations for each value J.

For J = 0 (pairing of nucleons with opposite momenta), the interaction between the surface deformation and the nucleon pair is zero in our approximation. In this case the nuclear energy depends on β and γ only through E_p and the potential energy of deformation. Since $C\beta^2/2$ does not depend on γ, but E_p obviously is weakly γ dependent, we shall simply neglect the dependence of E_p on γ in the investigation of the dependence of the nuclear energy on γ. Introducing $\epsilon = E(J) - E(0)$, where $E(0)$ is the nuclear energy for J = 0, we consider the equation for the determination of ϵ for J = 2:

$$\begin{vmatrix} {}^5/_2(a+b) - 6T\beta\cos\gamma - \varepsilon, & 6T\beta\sin\gamma + (a-b)\sqrt{3}/2 \\ 6T\beta\sin\gamma + (a-b)\sqrt{3}/2, & 6T\beta\cos\gamma + (a+b)/2 + 2c - \varepsilon \end{vmatrix} = 0.$$

Substituting (1.5), and expanding the determinant, we obtain the second degree equation

$$x^2 - \frac{9x}{2\sin^2 3\gamma} - \frac{9}{4} - \frac{27\cos 3\gamma}{4\sin^2 3\gamma}l + \frac{9}{2\sin^2 3\gamma} = 0, \qquad (2.3)$$

where

$$x = \varepsilon/(\hbar^2/B\beta^2), \qquad l = 4T\beta/(\hbar^2/B\beta^2), \qquad (2.4)$$

and $\hbar^2/B\beta^2$ is the energy of the first excited level of the nucleus.

For a rough estimate of the quantity l we set $T \approx 40$ Mev, $\hbar^2/B\beta^2 \sim 400$ kev, and $\beta = 0.2$. Then we find from (2.4) $l \approx 80$. As the deformation parameter β increases, the parameter l increases as $\sim\beta^3$.

In Fig. 1 we show the solutions of (2.4) as functions of γ for the values $l = 10$, 15, and 150.

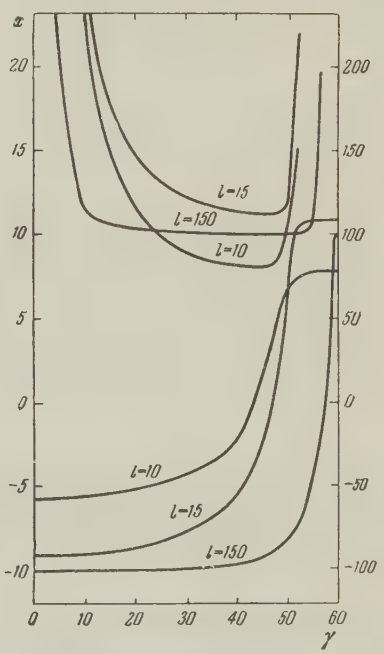

FIG. 1. Nuclear energy as a function of γ and of the quantity l which determines the coupling of the pair of outer nucleons (J = 2) with the nuclear surface deformation. The right hand scale gives the energy for $l = 150$.

In the case when the pair of outer nucleons is in a state with J = 4, the equation for the total nuclear energy has the form

$$x^3 - \frac{45x^2}{2\sin^2 3\gamma} - \left(39l^2 + 117l\frac{\cos 3\gamma}{\sin^2 3\gamma} - \frac{81}{\sin^4 3\gamma} - \frac{78}{\sin^2 3\gamma}\right)x$$
$$- 70l^3\cos 3\gamma + 5\left(42 - \frac{9}{2\sin^2 3\gamma}\right)l^2$$
$$+ 5l\left(81\frac{\cos 3\gamma}{\sin^4 3\gamma} + 42\right) - \frac{270}{\sin^4 3\gamma} - \frac{70}{\sin^2 3\gamma} = 0, \qquad (2.5)$$

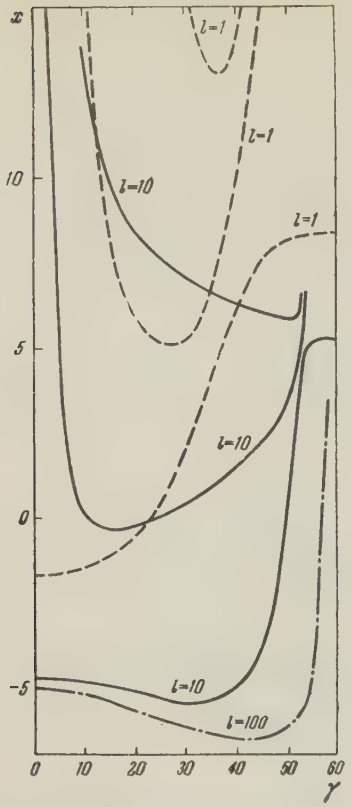

FIG. 2. The energy as a function of γ and l for $J = 4$. The value of the energy is to be multiplied by 10 for $l = 10$, and by 100 for $l = 100$.

where x and l are given by (2.4). Figure 2 shows the dependence of three roots of (2.5) on γ for the value $l = 10$ (solid curve). In the same figure we give the dependence of the lowest roots of (2.5) on γ for $l = 1$ (dotted curve) and $l = 100$ (dash-dotted curve).

3. THE SHAPE OF THE NUCLEUS AS A FUNCTION OF THE STRENGTH OF THE COUPLING BETWEEN THE PARTICLES AND THE NUCLEAR SURFACE

We apply the results of the preceding section to the problem of finding the equilibrium shape of the nucleus.

If each of the outer electrons is in the state $j = 1/2$, they can be paired only with the value $J = 0$. Such a nucleon pair has no effect on the shape of the nucleus.

If the nucleons are in the state $j = 3/2$, pairing can occur for $J = 0$ and $J = 2$. According to Fig. 1, the state with $J = 2$ has the lower energy. The minimum of the energy then corresponds to a nuclear shape with axial symmetry ($\gamma_0 = 0$). A pair of outer nucleons in states with $j = 3/2$ therefore does not destroy the axial symmetry of the nucleus.

If the nucleons are in the state $j = 5/2$, pairing can occur for $J = 0$, 2, and 4. A comparison of Figs. 1 and 2 shows that the minimal energy for a given value l corresponds to the largest possible angular momentum ($J = 4$). In this case the equilibrium shape of the nucleus for $l = 1$ also corresponds to an ellipsoid of revolution ($\gamma_0 = 0$). However, as l increases, the equilibrium shape of the nucleus corresponds to $\gamma_0 \neq 0$. As l reaches the value 10, the equilibrium shape of the nucleus corresponds to $\gamma_0 \approx 30°$. Here the deviation from axial symmetry reaches a maximum: the nucleus has a shape which is intermediate between the prolate and the oblate ellipsoids of revolution. For $l = 100$, $\gamma_0 \approx 42°$; for $l \to \infty$, $\gamma_0 \to 60°$, i.e., the shape of the nucleus becomes again axially symmetric. In the case of extremely strong coupling between the nucleons and the surface (β large) our results are therefore the same as those of Bohr and Ford.

If the nucleons in the pair are in a state with $j \geq 7/2$, we continue to observe the same behavior as in the case $j = 5/2$. The minimal energy corresponds to the largest possible value J. As the quantity l increases, the equilibrium value of γ changes from 0 to 60°. And the larger the value of J, the smaller is the value of l at which the deviation from axial symmetry begins.

Thus the effect of the interaction of the nucleon pair with the nuclear surface favors the pairing of the nucleons with the largest possible value of J. This effect therefore acts in the opposite direction of the effect of the attractive interaction of the fermions at the Fermi surface (in momentum space), which in certain cases (superconductivity of metals) leads to a coupling of the fermions with opposite spins and momenta. For sufficiently small l the competition between these effects favors the states with $J = 0$. For large l, on the other hand, the states with the largest possible J are more favored energetically. Owing to the Coulomb repulsion between protons, this occurs at smaller values of l for protons than for neutrons.

If the number of nucleons in the shell of angular momentum j corresponds to a closed shell, this state will have only one total angular momentum: $J = 0$.

Using the formulas of reference 6 and the experimental value for the ratio of the energy of the second excited level with spin 2 over the energy of the first excited level, we can determine the equilibrium value γ_0, up to the transformation $\gamma_0' \rightleftarrows 60 - \gamma_0$. This ambiguity arises from the fact that the position of the levels and the transition

probability do not depend on the sign of the quadrupole moment. These calculations lead to the result that the greatest deviation from axial symmetry ($30° > \gamma_0 > 27°$) occurs in the nuclei $_{46}Pd^{108}$, $_{46}Pd^{106}$, $_{48}Cd^{114}$, $_{52}Te^{122}$, $_{52}Te^{126}$, $_{78}Pt^{192}$, $_{78}Pt^{196}$, and several others in which the number of protons differs from the magic numbers 50 and 82 by two or four units. For $T = 40$ Mev and values β calculated from Coulomb excitation data, the parameter l, which determines the dependence of the nuclear ground state energy on γ, lies somewhere within the interval 20 to 80.

For nuclei with a ratio $E(2')/E(2)$ between the limits 2.9 and 20 (which corresponds to values of γ within the intervals $20° > \gamma_0 > 10°$ or $40° < \gamma_0 < 50°$), the parameter l has the value 300 to 600. Finally, for nuclei with nearly axial symmetry, for which $E(2')/E(2) > 23$ ($\gamma_0 < 10°$ or $\gamma_0 > 50°$), the parameter $l > 1000$. These experimental values are in qualitative agreement with the theoretical results.

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ON ANALYTIC PROPERTIES OF CASUAL COMMUTATORS

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A simple derivation is given of the integral representation for the causal commutator discovered by Jost and Lehmann[1] and generalized by Dyson,[2] which does not require the use of six dimensions. In the simpler cases (vertex part, two-particle matrix element) more detailed spectral formulas are found. On the basis of these formulas it is shown that the two-particle scattering amplitude — for real values of the energy in the center-of-mass system — is an analytic function of the square of the momentum transfer regular in the entire complex plane except for poles and cuts on the real axis.

1. Jost and Lehmann[1] discovered an integral representation for the matrix element of the causal commutator of two Heisenberg operators A and B

$$f(x) = \langle p,\ r\,|\,[A(x/2),\ B(-x/2)]\,|\,p',r'\rangle, \tag{1}$$

where $|p, r\rangle$ is the state characterized by a total momentum p_μ, and r denotes all other quantum numbers. However the representation for f(x) which they found was not manifestly covariant and was rigorously valid only in the symmetric case (A = B). The generalization to the nonsymmetric case was carried out in an invariant form by Dyson[2] by introducing six dimensions in momentum space.

The main results of the present work consist showing that the general four-parameter Jost-Lehmann-Dyson representation for f(x) can be made substantially more specific in the simpler cases and that it reduces to a two- and three-parameter representation respectively for the vertex part (when one of the states in (1) is the vacuum state and the other is a single-particle state) and the two-particle matrix element (when both states in (1) are single-particle states). Furthermore, a simple derivation of the general representation for f(x) is given, without recourse to six dimensions which, it seems to us, needlessly complicate the proofs.

The three-parameter representation for the two-particle matrix element permits a significant increase in the region of regularity of the scattering amplitude as a function of the momentum transfer Δ^2 as compared to the region found by Lehmann;[3] namely it is possible to show that the scattering amplitude (for real values of the energy in the center of mass system) is an analytic function of Δ^2 regular in the entire complex plane except for poles and cuts on the real axis.

It should be emphasized that we have not as yet succeeded in demonstrating the necessity of the limitations on the spectral function in the three-parameter representation of the matrix element for the scattering of two particles.

For the sake of simplicity we ignore the spin dependence of the matrix elements in the following.

2. Let us find a general representation for f(x) which makes use of the causality condition

$$f(x) = 0 \quad \text{for } x^2 \equiv x_0^2 - \mathbf{X}^2 < 0. \tag{2}$$

This condition allows f(x) to be written in the form

$$f(x) = \int_0^\infty d\lambda^2 f(x)\,\delta(x^2 - \lambda^2). \tag{3}$$

Let us introduce an auxiliary function

$$\varphi_{\lambda^2}(x) = f(x)\,\delta(x^2 - \lambda^2) \tag{4}$$

such that

$$f(x) = \int_0^\infty \varphi_{\lambda^2}(x)\,d\lambda^2. \tag{4'}$$

It is easy to see that the Fourier transform $\tilde{\varphi}_{\lambda^2}(q)$ of $\varphi_{\lambda^2}(x)$ satisfies the wave equation with a mass λ in q-space:

$$(\Box_q - \lambda^2)\,\varphi_{\lambda^2}(q) = 0,$$
$$\Box_q \equiv -\partial^2/\partial q_0^2 + \partial^2/\partial q_1^2 + \partial^2/\partial q_2^2 + \partial^2/\partial q_3^2, \tag{5}$$

The solution of Eq. (5) may be expressed in terms of the value and the normal derivative of $\varphi_{\lambda^2}(q)$ on an arbitrary space-life surface

$$\tilde{\varphi}_{\lambda^2}(q) = \int d\sigma_\alpha \left[\tilde{\varphi}_{\lambda^2}(u), \frac{\partial}{\partial u_\alpha}\right] \Delta(q-u,\ \lambda^2), \tag{6}$$

where $\Delta(q, \lambda^2)$ is the odd invariant function of Eq. (5);

$$\left[\varphi, \frac{\partial}{\partial x}\right]\psi \equiv \frac{\partial\varphi}{\partial x}\psi - \varphi\frac{\partial\psi}{\partial x};$$

and σ_α is an arbitrary three-dimensional space-like surface in q-space.

According to Eq. (4) the Fourier transform $\tilde{f}(q)$ of $f(x)$ is

$$\tilde{f}(q) = \int e^{-iqx} f(x)\, d^4x = \int_0^\infty \tilde{\varphi}_{\lambda^2}(q)\, d\lambda^2. \tag{7}$$

Inserting (6) into (7) we obtain the invariant Jost-Lehmann-Dyson representation in its most general form

$$\tilde{f}(q) = \int_0^\infty d\lambda^2 \int d\sigma_\alpha \left[\tilde{\varphi}_{\lambda^2}(u), \frac{\partial}{\partial u_\alpha}\right]\Delta(q-u, \lambda^2). \tag{8}$$

If we now write

$$\Delta(q-u, \lambda^2) = \int_0^\infty \varepsilon(q-u)\,\delta((q-u)^2 - \varkappa^2)\,\overline{\Delta}(x^2, \lambda^2)\, d\varkappa^2,$$

where

$$\overline{\Delta}(x^2, \lambda^2) = -\frac{2}{(2\pi)^4}\, P\int \frac{e^{-ipx}}{p^2 - \lambda^2}\, d^4p,$$

and set

$$\psi(\varkappa^2, u) = \int_0^\infty d\lambda^2\, \varphi_{\lambda^2}(u)\overline{\Delta}(\varkappa^2, \lambda^2),$$

then we obtain in place of Eq. (8) the following

$$\tilde{f}(q) = \int_0^\infty d\varkappa^2 \int d\sigma_\alpha \left[\psi(\varkappa^2, u)\frac{\partial}{\partial u_\alpha}\right] \times \varepsilon(q-u)\,\delta((q-u)^2 - \varkappa^2), \tag{8'}$$

i.e., the desired representation investigated in detail by Dyson.[2]* Choosing the surface $u_0 = 0$ in Eq. (8') we get the representation found by Jost and Lehmann.[1]

From Eq. (1) it follows that $\tilde{f}(q)$ vanishes in the region

$$P_0 - (m_1^2 + (\mathbf{q}-\mathbf{P})^2)^{1/2} \leqslant q_0 \leqslant (m_2^2 + (\mathbf{q}+\mathbf{P})^2)^{1/2} - P_0, \tag{9}$$

where $P = (p+p')/2$ and m_1 and m_2 are the masses of the lowest mass intermediate states $|n_1\rangle$ and $|n_2\rangle$ for which the matrix elements

$\langle p, r|A|n_1\rangle\langle n_1|B|p', r'\rangle$

and $\langle p, r|B|n_2\rangle\langle n_2|A|p', r'\rangle$

fail to vanish.

Dyson has shown that the representation (8') satisfies conditions (9) if and only if the function $\psi(\varkappa^2, u)$ vanishes everywhere except for the region**

$$\varkappa^2 \geqslant \max\{0, m_1 - |P+u|,\ m_2 - |P-u|\}$$
$$(P \pm u)^2 \geqslant 0. \tag{10}$$

We do not repeat that proof here. Let us only stress that in the derivation of Eq. (8') as well as in the deduction of the limitation (10) on $\psi(\kappa^2, u)$ there is no need whatsoever for introducing a space of six dimensions as was done in reference 2.

3. The general representation (8) or (8') depends on four parameters (λ^2 or κ^2 and the three components of the vector u). This is related to the fact that $\tilde{f}(q)$ depends on four quantities (q_0, q_1, q_2, q_3). In the derivation of (8) or (8') we have nowhere explicitly made use of the invariance properties of $f(x)$. We shall show below that it is possible to obtain more specific representations than (8) or (8') for the simpler matrix elements of the form (1) provided use is made of consequences of relativistic invariance. In general $f(x)$ may be expressed in terms of the invariants x^2, $p\cdot x$, $p'\cdot x$, etc.:† $f(x) \equiv f(x^2, p\cdot x, p'\cdot x, \ldots)$ where the dots denote all other possible invariants beside x^2. Consequently Eq. (3) may be rewritten in the form‡

$$f(x) = \int_0^\infty d\lambda^2 f(\lambda^2, px, p'x\ldots)\,\delta(x^2 - \lambda^2). \tag{3'}$$

We now use the relation***

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*The function $\psi(\varkappa^2, u)$ is related to the function $F(u, s)$ of Eq. (30), reference 2, by $\psi(\varkappa^2, u) = \partial F(u, \varkappa^2)/\partial\varkappa^2$.

**We note that the proof of necessity of the condition (10) in Dyson's work cannot be considered complete. In the proof, Dyson introduced the concept of admissible hyperboloids $(q-u)^2 - \varkappa^2 = 0$, corresponding to values of u and $\varkappa^2$ satisfying condition (10), and showed, using theorems of Jost and Lehmann[1] that for any twice inadmissible hyperboloid tha corresponding value of $\psi(\varkappa^2, u)$ is zero. A twice inadmissible hyperboloid is a hyperboloid whose both sheets (lower and upper) are in q-space inside region (9). However, as is easy to see by drawing an appropriate figure, a majority of the points u and $\varkappa^2$ that lie outside the region (10) correspond to hyperboloids $(q-u)^2 - \varkappa^2 = 0$ that have only one sheet inadmissible, i.e. the majority of the inadmissible hyperboloids is not twice inadmissible; and for just such hyperboloids Dyson did not show that the corresponding values of $\psi(\varkappa^2, u)$ are zero (this reservation applies equally to the cases of symmetric and non-symmetric regions).

†Strictly speaking the function $f(x)$ is an invariant only when multiplied by factors of the type $(2p_0)^{1/2}$; we ignore these factors.

‡Equation (3) is, of course, not unique. In particular, one may replace x_0 by $\pm(x^2 + \lambda^2)^{1/2}$ everywhere in $f(\lambda^2, p\cdot x\ldots)$, such a substitution leads to the Jost-Lehmann[1] formula [see (14')].

***This identity is easily verified starting from the parametric representation.

$$\int_{-\infty}^{+\infty} dx^2\, \bar{\Delta}(x^2, \varkappa^2)\, \bar{\Delta}(x^2, \lambda^2)$$

$$= \int_0^{\infty} \bar{\Delta}(x^2, \varkappa^2)\, \bar{\Delta}(x^2, \lambda^2)\, dx^2 = (2\pi)^{-2}\, \delta(x^2 - \lambda^2). \qquad (11)$$

Here $\bar{\Delta}(x^2, \kappa^2) = \epsilon(x_0)\, \Delta(x, \kappa^2)$, where $\Delta(x, \kappa^2)$ is the commutator function with mass κ, and $\epsilon(x_0)$ is the sign function. Introducing (11) into (3′) and denoting

$$\rho(x^2, px, \ldots) = (2\pi)^2 \int_0^{\infty} d\lambda^2 f(\lambda^2, px, \ldots)\ \varepsilon(x)\ \bar{\Delta}(x^2, \lambda^2), \qquad (12)$$

we find

$$\tilde{f}(x) = \int_0^{\infty} dx^2 \rho(x^2, px, \ldots)\, \Delta(x, \varkappa^2). \qquad (13)$$

From here it follows that the Fourier transform $\tilde{f}(q)$ is

$$\tilde{f}(q) = \int_0^{\infty} dx^2 \int d^4u\, \Phi(x^2, u)\ \ \varepsilon(q - u)\, \delta((q - u)^2 - x^2), \qquad (14)$$

where

$$\Phi(x^2, u) = \frac{-i}{(2\pi)^3} \int \rho(x^2, px \ldots)\, e^{-iux} d^4x. \qquad (14')$$

The expression (14) for $\tilde{f}(q)$ coincides in form with the five parameter representation obtained by Dyson (formula (49) of reference 2, theorem "c"), which also follows directly from Eq. (8).

For purposes of application (see reference 3) the derivation of restrictions of the type (10) on the spectral function $\Phi(\kappa^2, u)$ appearing in (14) is of greatest importance. That these restrictions are sufficient is obvious. That they are necessary has not yet been shown,* however one can hardly doubt the validity of the conditions (10) for $\Phi(\kappa^2, u)$. In particular it is clear from physical considerations that the only contributions to $\tilde{f}(q)$ come from the admissible hyperboloids $(q-u)^2 - \kappa^2 = 0$, since only they correspond to possible physical processes for which the matrix element of $\tilde{f}(q)$ does not vanish.

Let us apply formula (14) to some simple matrix elements.

$$\bar{\Delta}(x^2, \varkappa^2) = (2\pi)^{-2} \int_{-\infty}^{+\infty} d\alpha \exp[i\alpha x^2 + ix^2/4\alpha].$$

All relations and integrals encountered in this paper are to be interpreted in the distribution-theory sense.

*In this connection see footnote **, page 1067.

A. Vertex Part

The antihermitian part of the vertex function is expressed in terms of a matrix element of the form

$$f(x) = \langle 0 | [A(x/2),\ B(-x/2)] | p \rangle, \qquad (15)$$

where $|0\rangle$ and $|p\rangle$ are the vacuum and one-particle states respectively. In this case $f(x)$ depends on three invariants: x^2, $p \cdot x$, and p^2,* of which only the first two are x-dependent. Consequently a two-parameter representation for $f(q)$ should exist instead of the four- or five-parameter representations given by Eqs. (8′) and (14). Indeed the function $\rho(\kappa^2, p \cdot x, p^2)$ in Eq. (13) may be written in the form

$$\rho(x^2, px, p^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha(px)}\ \tilde{\rho}(x^2, \alpha, p^2)\, d\alpha, \qquad (16)$$

where α is some scalar parameter. From this, taking into account Eq. (14′), we obtain

$$\Phi(x^2, u) = (-i) \int_{-\infty}^{+\infty} \tilde{\rho}(x^2, \alpha, p^2)\, \delta(u - \alpha p)\, d\alpha. \qquad (17)$$

We now insert (17) into (14), integrate over d^4u and obtain

$$\tilde{f}(q) = (-i) \int_0^{\infty} dx^2 \int_{-\infty}^{+\infty} d\alpha\, \tilde{\rho}(x^2, \alpha, p^2)$$

$$\times\, \varepsilon(q - \alpha p)\, \delta((q - \alpha p)^2 - x^2). \qquad (18)$$

For the Fourier transform of the retarded commutator

$$f_R(q) = \int \theta(x) f(x) \exp(-iqx)\, d^4x,$$

we have the following spectral representation

$$f_R(q) = \frac{1}{2\pi} \int_0^{\infty} dx^2 \int_{-\infty}^{+\infty} \frac{d\alpha\, \tilde{\rho}(x^2, \alpha, p^2)}{((q - \alpha p)^2 - x^2)},$$

$$q_0 \to q_0 - i\varepsilon. \qquad (19)$$

The restrictions on the spectral function are the following: $\tilde{\rho}$ is different from zero only in the region $(\mathbf{p} = 0;\ p_0 = m;\ m_1 \geq m_2)$

$$-\tfrac{1}{2} + (m_1 - \varkappa)/m \leqslant \alpha \leqslant \tfrac{1}{2} - (m_2 - \varkappa)/m \quad \text{if } \varkappa \leqslant m_2$$

$$-\tfrac{1}{2} + (m_1 - \varkappa)/m \leqslant \alpha \leqslant \tfrac{1}{2} \quad \text{if } m_2 \leqslant \varkappa \leqslant m_1$$

$$-\tfrac{1}{2} \leqslant \alpha \leqslant \tfrac{1}{2} \quad \text{if } \varkappa > m_1$$

Thus in the case of the vertex part, which, except for unimportant factors is equal to $f_R(q)$, it is possible to derive a two-parameter (α and κ^2)

*The function $\varepsilon(x_0)$ may always be written as $\varepsilon(p \cdot x)$ because p is a timelike vector with $p_0 \geqslant 0$.

representation (18) and (19) instead of (14), by using relativistic invariance.

B. Two Particle Matrix Element

In this case

$$f(x) = \langle p' | [A(x/2),\ B(-x/2)] | p' \rangle \qquad (20)$$

and it is related to the matrix element for the scattering of two particles.

Repeating the considerations of section A gives*

$$\tilde{f}(q) = i \int_{-\infty}^{+\infty} d\alpha\, d\beta \int_0^{\infty} d\varkappa^2\, \tilde{\rho}(\varkappa^2, \alpha, \beta, Q^2)\, \varepsilon(q - \alpha P - \beta Q) \times \delta((q - \alpha P - \beta Q)^2 - \varkappa^2), \qquad (21)$$

$$f_R(q) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\alpha\, d\beta \int_0^{\infty} d\varkappa^2 \frac{\tilde{\rho}(\varkappa^2, \alpha, \beta, Q^2)}{(q - \alpha P - \beta Q)^2 - \varkappa^2}, \quad q_0 \to q_0 - i\varepsilon, \qquad (22)$$

where

$$P = (p + p')/2; \quad Q = (p - p')/2; \quad PQ = 0,$$

$$\text{if } p^2 = p'^2 = m^2; \quad Q^2 = -\Delta^2;$$

Δ^2 — the momentum transfer; the region in which the $\tilde{\rho}$ of Eqs. (21) and (22) is different from zero (in the frame $\mathbf{P} = 0$) is given as follows:

$$-1 + \left(\left(\frac{m_1 - \varkappa}{P_0}\right)^2 + \beta^2 \frac{\Delta^2}{P_0^2}\right)^{1/2} \leqslant \alpha \leqslant 1 - \left(\left(\frac{m_2 - \varkappa}{P_0^2}\right)^2 + \beta^2 \frac{\Delta^2}{P_0^2}\right)^{1/2}, \quad \varkappa \leqslant m_2,$$

$$-1 + \left(\left(\frac{m_1 - \varkappa}{P_0}\right)^2 + \beta^2 \frac{\Delta^2}{P_0^2}\right)^{1/2} \leqslant \alpha \leqslant 1 - \left|\frac{\beta\Delta}{P_0}\right|, \quad m_2 \leqslant \varkappa \leqslant m_1,$$

$$-1 + |\beta\Delta/P_0| \leqslant \alpha \leqslant 1 - |\beta\Delta/P_0|, \quad \varkappa > m_1. \qquad (23)$$

Consequently the two particle matrix element has a three-parameter representation (α, β, and κ^2).

Let us note an important property of Eqs. (18), (19), (21), and (22). In the limit as p and p′ tend to zero they automatically go over into the well known formulas of Kallen-Lehmann[4] for the one-particle Green's function.

For the study of analytic properties of the matrix element for nucleon-meson scattering as a function of momentum transfer Δ^2, it is more convenient to consider instead of (20) a retarded commutator of the form

$$f_R(x) = i\theta(x) \langle 0 | [\eta(x/2),\ j(-x/2)] | p;\ k \rangle,$$

where j is the nucleon current and η is the right hand side of the equation for the nucleon field operators; p and k are the initial momenta of the nucleon and meson. The retarded scattering amplitude $f_R(q = (p' - k')/2)$ is equal, in this case (up to unimportant factors), to the Fourier transform of $f_R(x)$ and can be expressed in the form (22) with a different spectral function $\tilde{\rho}_1$:

$$f_R(q) = \frac{1}{2\pi} \int \frac{d\alpha\, d\beta\, d\varkappa^2 \tilde{\rho}_1(\varkappa^2, \alpha, \beta, p, k)}{[(p' - k') - \alpha(p + k) - \beta(p - k)]^2/4 - \varkappa^2}. \qquad (24)$$

The limitations on $\tilde{\rho}_1$ of the type (23) may be easily obtained from conditions (10) by setting $u = [\alpha(p+k) + \beta(p-k)]/2$ in the latter.

In the center of mass system ($p+k = 0$; $(p+k)^2 = w^2$; $m_1 = m + \mu$; $m_2 = 3\mu$) the entire dependence of $f_R(q)$ in Eq. (24) on Δ^2 is in the denominator. Carrying out considerations fully analogous to those of Lehmann[3] one can show that the scattering amplitude $f_R(q) \equiv f_R(w^2, \Delta^2)$ is regular in Δ^2 everywhere except for the following region on the real axis

$$1 - 2\Delta^2/k^2 \geqslant [1 + 8\mu^3(2m + \mu)/k^2(w^2 - (m - 2\mu)^2)]^{1/2}. \qquad (25)$$

To find the region in which the imaginary part of $f_R(w^2, \Delta^2)$ is regular in Δ^2 it is necessary to use the general formula (14) (see reference 3) because expressions of the form (24) turn out to be insufficient.

*This formula was derived by a different method by V. D. Skarzhinskiĭ (Thesis, Moscow State University, 1957).

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ON THE DEPENDENCE OF THE ANGLE BETWEEN THE DIRECTION OF MOTION OF SHOWER PARTICLES AND THE AXIS OF THE SHOWER ON THE DISTANCE FROM THE AXIS

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The dependence of the mean angle between the motion of particles and the axis of a shower on the distance from the axis is calculated. The case considered is that of an electron-photon shower averaged over the depth, with no account taken of ionization losses. The results of the calculation are compared with experiment.

PAPERS on the three-dimensional cascade theory of showers have dealt with either the spatial distribution functions of the particles, integrated over the angle variables, or the angular distribution functions, integrated over a plane perpendicular to the axis of the shower.[1] At the present time, however, in connection with the study of high-energy electron-photon showers in photographic plates, and also in the study of the soft component in the cores of extensive atmospheric showers one needs a knowledge of the complete spatial and angular distribution function of the particles. For the correct analysis of the experimental data it is particularly important to know the angular distribution function of the particles at a prescribed distance from the axis of the shower. This problem is extremely complicated mathematically, and here there is clearly no hope of getting an analytic solution. Therefore it is necessary to look for convenient approximate methods for solving the problem. It seems to us that in this case the method of moments can be very useful. Another possible approach is the numerical solution of the problem with high-speed electronic computing machines. But because of certain peculiarities of the electromagnetic cascade process — the energy spectrum of the particles goes as E^{-s} and the multiplicity increases rapidly with the depth — the solution of problems of this type encounters grave difficulties, and there has so far not been a single electronic-computer calculation of the three-dimensional development of a shower.

In the present paper we take the first step of calculating the average angle $\bar{\theta}_x(E, x)$ between the axis of the shower and the motion of particles having an energy E and present at a given distance x from the axis. A knowledge of the function $\bar{\theta}_x(E, x)$ given by the cascade theory will make it possible to proceed with more assurance in taking into account the effects of nuclear processes in the development of a given shower, if the nature of the primary particle that produced the shower is not evident. We shall consider the "equilibrium" value of $\theta_x(E, x)$, i.e., the average obtained by integration over the depth of development of the shower. Since this function is primarily of physical interest for $x \ll 1$, we neglect ionization losses. It can be shown that in an electromagnetic cascade shower the number of particles with energy $E < 10^8$ ev is small out to distances $\tilde{x} = Ex/E_s \lesssim 0.1$ (here x is expressed in terms of t as a unit, and $E_s = 21$ Mev).

Let the position of the particle in the plane perpendicular to the axis of the shower be given by the coordinates x and y. The direction of motion of the particle is given by the angles θ_x and θ_y in two mutually perpendicular planes with their intersection parallel to the axis of the shower. Let $P(E_0, E, x, y, \theta_x, \theta_y)\,dEdxdyd\theta_x d\theta_y$ be the number of electrons with energy in the range dE, position in the area $dxdy$, and motion in the range of angles $d\theta_x d\theta_y$ in a shower produced by a primary particle of energy E_0; and let $\Gamma(E_0, E, x, y, \theta_x, \theta_y)\,dEdxdyd\theta_x d\theta_y$ be the analogous number of photons. We treat the scattering as multiple, in the Landau approximation.[2] The calculation is made in the small-angle approximation, i.e., $\cos\theta$ is replaced by 1 and $\sin\theta$ by θ. We neglect the backward current of particles through the boundary of the absorber at $t = 0$.[3]

For the case in which an electron of energy E_0 is incident vertically on the boundary of the layer of material at $t = 0$, we integrate the fundamental

equations of three-dimensional cascade theory with respect to t from 0 to ∞ and with respect to y and θ_y from $-\infty$ to ∞. We multiply the equations so obtained by θ_x^n and integrate them with respect to θ_x from $-\infty$ to ∞ and with respect to x from $-\infty$ to x_0. We then have:

$$\int_{-\infty}^{\infty} P(E_0, E, x_0, \theta_x)\,\theta_x^{n+1} d\theta_x = L_1[P_{0n}(E_0, E, x_0), \Gamma_{0n}(E_0, E, x_0)] + \frac{E_s^2}{4E^2} n(n-1) P_{0,\,n-2}(E_0, E, x_0) - \delta(E_0 - E)\,\delta_{n0};$$

$$\int_{-\infty}^{\infty} \Gamma(E_0, E, x_0, \theta_x)\,\theta_x^{n+1} d\theta_x = L_2[P_{0n}(E_0, E, x_0), \Gamma_{0n}(E_0, E, x_0)]. \quad (1)$$

Here L_1 and L_2 are integral operators that give the effects of radiative deceleration and pair production; $\delta(E_0 - E)$ is the δ function; and δ_{n0} is the Kronecker symbol. We have introduced the notations

$$P_{0n}(E_0, E, x_0) = \int_{-\infty}^{x_0} dx \int_{-\infty}^{\infty} P(E_0, E, x, \theta_x)\,\theta_x^n d\theta_x;$$

$$\Gamma_{0n}(E_0, E, x_0) = \int_{-\infty}^{x_0} dx \int_{-\infty}^{\infty} \Gamma(E_0, E, x, \theta_x)\,\theta_x^n d\theta_x, \quad (2)$$

where

$$P(E_0, E, x, \theta_x) = \iint_{-\infty}^{\infty} P(E_0, E, x, y, \theta_x, \theta_y)\,dy d\theta_y;$$

$$\Gamma(E_0, E, x, \theta_x) = \iint_{-\infty}^{\infty} \Gamma(E_0, E, x, y, \theta_x, \theta_y)\,dy d\theta_y. \quad (3)$$

The quantities in which we are interested are

$$\overline{\theta}_x^P(E, x_0) = \int_{-\infty}^{\infty} P(E_0, E, x_0, \theta_x)\,\theta_x d\theta_x \Big/ \int_{-\infty}^{\infty} P(E_0, E, x_0, \theta_x)\,d\theta_x;$$

$$\overline{\theta}_x^{\Gamma}(E, x_0) = \int_{-\infty}^{\infty} \Gamma(E_0, E, x_0, \theta_x)\,\theta_x d\theta_x \Big/ \int_{-\infty}^{\infty} \Gamma(E_0, E, x_0, \theta_x)\,d\theta_x. \quad (4)$$

In order to find them, we must examine the system (1) for $n_4 = 0$. Then, using the explicit expressions for the operators L_1 and L_2,[4] we have

$$\int_{-\infty}^{\infty} P(E_0, E, x_0, \theta_x)\,\theta_x d\theta_x = 2\int_0^1 \Gamma_{00}\left(E_0, \frac{E}{u}, x_0\right)\varphi_0(u)\,\frac{du}{u} - \int_0^1 \left[P_{00}(E_0, E, x_0) - \frac{1}{1-v} P_{00}\left(E_0, \frac{E}{1-v}, x_0\right)\right]\varphi_0(v)\,dv - \delta(E_0 - E);$$

$$\int_{-\infty}^{\infty} \Gamma(E_0, E, x_0, \theta_x)\,\theta_x d\theta_x = \int_0^1 P_{00}\left(E_0, \frac{E}{v}, x_0\right)\varphi_0(v)\,\frac{dv}{v} - \sigma_0 \Gamma_{00}(E_0, E, x_0). \quad (5)$$

Here

$$P_{00}(E_0, E, x_0) = \int_{-\infty}^{x_0} P(E_0, E, x)\,dx,$$

$$\Gamma_{00}(E_0, E, x_0) = \int_{-\infty}^{x_0} \Gamma(E_0, E, x)\,dx.$$

For $P(E_0, E, x)$ and $\Gamma(E_0, E, x)$ we can obtain the following expressions:

$$P(E_0, E, x) = 2\int_x^{\infty} \frac{P(E_0, E, r)\,r dr}{\sqrt{r^2 - x^2}},$$

$$\Gamma(E_0, E, x) = 2\int_x^{\infty} \frac{\Gamma(E_0, E, r)\,r dr}{\sqrt{r^2 - x^2}}. \quad (6)$$

The functions $P(E_0, E, r)$ and $\Gamma(E_0, E, r)$ can be represented in the form

$$P(E_0, E, r) = P_{\text{long}}(E_0, E)\,P_r(Er/E_s),$$

$$\Gamma(E_0, E, r) = \Gamma_{\text{long}}(E_0, E)\,\Gamma_r(Er/E_s), \quad (7)$$

where

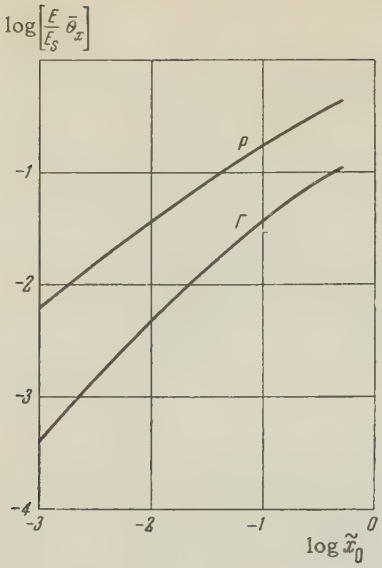
$$P_{\text{long}}(E_0, E) = P(E_0, E)\,E^2/E_s^2,$$

$$\Gamma_{\text{long}}(E_0, E) = \Gamma(E_0, E)\,E^2/E_s^2. \quad (8)$$

$P(E_0, E)$ and $\Gamma(E_0, E)$ are the equilibrium spectra of electrons and photons, respectively. The functions P_r and Γ_r are normalized in the following way:

$$\int_0^{\infty} P_r(\tilde{x})\,\tilde{x} d\tilde{x} = \int_0^{\infty} \Gamma_r(\tilde{x})\,\tilde{x} d\tilde{x} = 1. \quad (9)$$

For the numerical calculations of $\overline{\theta}_r(E, x_0)$ we obtained values of the functions $P_r(x)$ and $\Gamma_r(x)$ by the method of moments explained in reference 5. By substituting all the required values in Eq. (5) and carrying out the rather lengthy calculation one can get convenient formulas for the computation of $\overline{\theta}_x^P(E, x_0)$ and $\overline{\theta}_x^{\Gamma}(E, x_0)$; these contain single integrations and two double integrations that can be computed relatively easily. The results of the computation are shown in the diagram. It is interesting to compare the values of the mean angle so obtained with experiment. N. L. Grigorov and M. A. Kondrat'eva have obtained the corresponding experimental values in a study of an electron shower caused by a primary electron or photon or energy $E_0 \approx 10^{13}$ ev. The table shows the experimental and calculated values of the mean angle for various distances from the axis. It can be seen from the table that the calculated values of $\overline{\theta}_x$ agree well with the experimental values. We must take into account the fact that the authors mentioned measured the angles $\overline{\theta}$, not the projections $\overline{\theta}_x$. Knowing the angular distribution of the shower particles at a given distance from the axis, we can



Dependence of the mean angle $\bar{\theta}_x$ (in radians) made by paths of particles with the axis of the shower on the distance from the axis. Abscissas are values of the logarithm of the distance $\tilde{x}_0 = Ex_0/E_s$. Ordinates are values of the logarithm of the mean angle with the axis for electrons (curve P) and photons (curve Γ), in both cases multiplied by E/E_s.

show that $\bar{\theta}_x \approx \bar{\theta}_{exp}/1.6$. Furthermore the experimental errors in these measurements of $\bar{\theta}$ were 20 to 30 percent. The agreement of the calculated and experimental values of $\bar{\theta}_x$ enables us to conclude with assurance that the shower studied by Grigorov and Kondrat'eva was a pure electron-photon shower. We can at any rate be sure that the part played by nuclear processes in the development of this shower was a small one.

r/t	0.01	0.04	0.1	0.2
$\bar{\theta}^\circ_{x\,calc}$	0.7	2.1	4.6	7.9
$\bar{\theta}^\circ_{exp}$	1.2	4.2	9	11.5
$\bar{\theta}^\circ_{x\,exp} \approx \bar{\theta}^\circ_{exp}/1.6$	0.75	2.6	5.6	7.2

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RADIATIVE CAPTURE OF POLARIZED μ^- MESONS BY NUCLEI

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The correlation between the direction of the μ-meson spin and the direction of emission of the photon in the radiative capture of μ mesons by nuclei is considered, taking into account the interaction between the μ meson and the nuclear spin (hyperfine splitting). The analysis is carried out for nuclei of arbitrary spin J.

IT is known[1] that, in the case of radiation capture of a polarized μ^- meson (internal bremsstrahlung), there is correlation between the emission of the photon and the direction of the spin of the μ^- meson when parity is not conserved. An experimental observation of the asymmetry of the emission of the radiative photon makes it possible to establish the type of the interaction between the μ meson and the nucleus (A-V or T-S), if it is known whether the neutrino is right-handed or left-handed.

In the paper by Huang, Yang, and Lee,[1] the result of a calculation of the radiation capture of polarized μ^- mesons by protons is presented for the case of a two-component neutrino. As has been shown by Gershteĭn and Zel'dovich[2] μ^- mesons are fully depolarized in hydrogen because of the transition of μ^- mesons from one proton to another. It is therefore necessary to consider the capture of μ^- mesons by heavy nuclei. In addition, for nucleus with spin different from zero, it is necessary to take into account the depolarization of μ mesons due to interaction of the spins of the μ meson and the nucleus, and to consider separately two possible states with $F = J^{\pm 1/2}$ (hyperfine structure). This has first been shown by Bernstein, Lee, Yang, and Primakoff[3] in an analysis of the normal capture of μ^- mesons by nuclei. In view of the conservation of spin of the system nucleus—μ meson, this depolarization of mesons leads to a partial polarization of the nuclei. In view of this fact, taking into account the superfine structure will, apart from changing the numerical results, lead to a dependence of the studied correlation on the product of the interaction constants $C_A C_V$ and $C_S C_T$: i.e., it becomes possible to study the contribution of interference terms in the μ-meson capture.

CORRELATION IN THE PRESENCE OF HYPERFINE STRUCTURE

Let a μ meson polarized along the z axis be captured by the K orbit of the nucleus producing a mesic atom. Let the projection of the nuclear spin on the z axis be M, and the absolute value of the spin be J.

Since the frequency corresponding to the hyperfine splitting of the mesic atom levels is, in all cases, much larger than the inverse lifetime of the μ meson, one can assume that the mesic atom is in a state described by the wave function

$$\Psi = \sum_F C(F, F_z)\, \Phi_{F, F_z} e^{iE_F t}. \qquad (1)$$

where F is the total moment of the mesic atom which assumes the two values $F = J \pm 1/2$; $F_Z = M \pm 1/2$ is the projection of the total moment on the z axis; $\Phi F, F_Z$ is the wave function of the system μ^- meson-nucleus in a state with determined F and F_Z; E_F is the energy of the hyperfine structure level; and $C(F, F_Z) = \left(\begin{matrix} J\,1/2\,F \\ M\,1/2\,F_Z \end{matrix}\right)$ is the Clebsch-Gordan coefficient (Wigner's notation) corresponding to the condtion that, for $t = 0$, the wave function Ψ is a product of the wave functions of the nucleus with a given M and of the μ^- meson with a given projection s_Z ($\psi_M \psi_{1/2}$).

From Eq. (1) we obtain the probability of μ meson capture averaged over time:*

$$W = \sum_F C^2(F, F_z)\, W_{F, F_z}, \qquad (2)$$

where W_{F, F_Z} is the probability of capture in the Φ_{F, F_Z} state of the mesic atom. Since

*In the above, we do not take into account the fact that the lifetime of the mesic atom with respect to the simple capture of μ^- mesons is different for states with $F = J + 1/2$ and $F = J - 1/2$, which leads to a change of the statistical weight of W_{F, F_z}. In addition, we do not consider the possibility of transition from states with $F = J + 1/2$ to those with $F = -1/2$.

$$\Phi_{F,F_z}=\begin{pmatrix} J & 1/2 & F\\ M & 1/2 & F_z\end{pmatrix}\left|\psi_M\varphi_{1/2}\right\rangle+\begin{pmatrix} J & 1/2 & F\\ M+1 & -1/2 & F_z\end{pmatrix}\left|\psi_{M+1}\varphi_{-1/2}\right\rangle,$$

$$\begin{aligned}W_{F,F_z}&\sim|(\chi_f|\Omega|\Phi_{F,F_z})|^2\\ &=\begin{pmatrix} J & 1/2 & F\\ M & 1/2 & F_z\end{pmatrix}^2|(\chi_f|\Omega|\psi_M\varphi_{1/2})|^2\\ &+\begin{pmatrix} J & 1/2 & F\\ M+1 & -1/2 & F_z\end{pmatrix}^2|(\chi_f|\Omega|\psi_{M+1}\varphi_{-1/2})|^2\\ &+\begin{pmatrix} J & 1/2 & F\\ M & 1/2 & F_z\end{pmatrix}\begin{pmatrix} J & 1/2 & F\\ M+1 & -1/2 & F_z\end{pmatrix}\\ &\times[(\chi_f|\Omega|\psi_M\varphi_{1/2})^*(\chi_f|\Omega|\psi_{M+1}\varphi_{-1/2})\\ &+(\chi_f|\Omega|\psi_M\varphi_{1/2})(\chi_f|\Omega|\psi_{M+1}\varphi_{-1/2})^*],\end{aligned}\qquad(3)$$

where χ_f is the wave function of the final state, and Ω is the Hamiltonian of the considered interaction. The probabilities $\omega_{M,\,s_z}\sim|\chi_f|\Omega|\psi_{M s_z})|^2$ are calculated by the usual method and are given in the Appendix (A1). It will be shown that if $\omega_{M,\,s_z}$ is known for any values of M and s_z, then the probability W_{F,F_z} can be found without calculating the superposition terms in the expression (3).

In calculating W_{F,F_z}, we are not interested in the spin and momentum of the recoil nucleus, the direction and polarization of the neutrino, and the polarization of the photon,* i.e. we carry out a summation and integration over the corresponding values. After this, the final state will be characterized by one vector only: the momentum of the photon k. The μ-meson capture probability in a mesic atom with a moment F, can therefore depend only on the modulus of $|k|$ and $|F|$ and on the different powers of the product kF.

Consequently, the probability W_{F,F_Z} is of the form

$$\begin{aligned}W_{F,F_z}&\sim|(\Phi_{F,F_z}|a_F+b_F(Fk)+c_F(Fk)^2|\Phi_{F,F_z})|^2\\ &=a_F+b_FF_zk_z+{}^1/_2c_F\{[F(F+1)\\ &-F_z^2]k^2+[3F_z^2-F(F+1)]k_z^2\}.\end{aligned}\qquad(4)$$

where a_F, b_F and c_F are coefficients depending only on F and k.

In the first approximation of perturbation theory the probability W_{F,F_Z} is proportional to k^2, i.e. $a_F\sim k^2$, $b_F\sim|k|$, c_F is independent of k. A power of (kF) not higher than two should therefore be taken in Eq. (4).

Equation (4) determines the dependence of the probability on F_Z. To find the coefficients a_F, b_F, and c_F, a particular case $F_z=F=J+\frac{1}{2}$ has been considered. Since

$$\begin{pmatrix} J & 1/2 & J+1/2\\ M+1 & -1/2 & J+1/2\end{pmatrix}=0,$$

it follows from Eq. (3) that

$$W_{J+1/2,\,J+1/2}=\omega_{J,\,1/2},\qquad(5)$$

and $\omega_{j,\,\frac{1}{2}}$ is the known probability (see Appendix).

Equation (5) determines a_F, and c_F for $F=J+\frac{1}{2}$. If $W_{J+\frac{1}{2},\,F_Z}$ is known, we find $W_{F,\,F_Z}$ for the case $F=J-\frac{1}{2}$ from the same expression (3), excluding the superposition term,

$$W_{J-1/2,\,F_z}=\omega_{M,\,1/2}+\omega_{M+1,\,-1/2}-W_{J+1/2,F_z}.$$

The final expression for $W_{F,\,F_Z}$ is given in the Appendix.

Taking into account Eqs. (2) and (4), and averaging over F_Z, we obtain the correlation function in the form

$$\overline{W}\sim1+\beta\cos\theta,\qquad(6)$$

where θ is the angle between the direction of the original spin of the μ meson and the momentum of the photon; the value of β is given in the Appendix.

For the case of a two-component neutrino ($C'=\pm C$), we have

$$\begin{aligned}\xi\beta=&\pm\frac{1}{(2J+1)^2}\Big\{|M_F|^2|C_V|^2\\ &+\Big[1+\frac{4}{3}J(J+1)\Big]|M_{GT}|^2|C_A|^2\\ &+f(J)\frac{4}{3}J(J+1)\Big|M_FC_V-\frac{M_{GT}C_A}{\sqrt{J(J+1)}}\Big|^2\Big\}\\ &\mp\frac{1}{(2J+1)^2}\Big\{|M_F|^2|C_S|^2\\ &+\Big[1+\frac{4}{3}J(J+1)\Big]|M_{GT}|^2|C_T|^2\\ &+f(J)\cdot\frac{4}{3}J(J+1)\Big|M_FC_S-\frac{M_{GT}}{\sqrt{J(J+1)}}C_T\Big|^2\Big\},\end{aligned}\qquad(7)$$

where the upper sign should be taken for $C'=C$ and the lower for $C'=-C$.

$$\xi=|M_F|^2(|C_S|^2+|C_V|^2)+|M_{GT}|^2(|C_A|^2+|C_T|^2).$$

and f(J) equals $J_f=J,\ J+1,\ J$ for $J_f=J,\ J-1,\ J+1$ respectively, where J_f is the spin of the recoil nucleus.

For J = 0, we have the results which can be obtained without taking the superfine structures into account.

$$\beta=\pm(|C_A|^2-|C_T|^2)/(|C_A|^2+|C_T|^2),\quad\text{for } J_f=1$$
$$\beta=\pm(|C_V|^2-|C_S|^2)/(|C_V|^2+|C_S|^2),\quad\text{for } J_f=0.$$

*As is well known,[4] in radiation capture, if the neutrino is a two-component one, the photon is totally circularly polarized.

The experiments of Goldhaber, Grodzins, and Sunyar[7] indicate that, for electronic capture, the neutrino is left-handed. From the conservation of the lepton charge and from the decay mode $\mu^- = e^- + \nu + \bar{\nu}$ it follows that the leptic charge of the μ^- meson is the same as of the electron (in contrast to the assumption of Zel'dovich[5]). One should therefore expect that a left-handed neutrino is emitted in μ^- capture also. The change of the sign of β even in the case where recoil nuclei with various J_f are produced, makes it then possible to determine without ambiguity whether we are dealing with the V-A or S-T type of interaction.

A direct determination of the helicity of the neutrino in μ decay can be carried out by studying the correlation between the direction of emission of the recoil nucleus and the spin direction of the polarized μ meson, as has been recently shown by Treiman.[6]

In conclusion, the author would like to express his gratitude to Ya.B. Zel'dovich for his interest in the work and for valuable remarks.

APPENDIX

We present here certain intermediate and final formulae:

$$\omega_{M,s_z} \sim \xi + (Es_z + GM)\cos\theta + Hs_zM\cos^2\theta,$$

where

$$\xi = {}^1/_2 |M_F|^2 (|C_S|^2 + |C_S'|^2 + |C_V|^2 + |C_V'|^2) + {}^1/_2 |M_{GT}|^2 (|C_A|^2 + |C_A'|^2 + |C_T|^2 + |C_T'|^2),$$

$$E = 2|M_F|^2 \operatorname{Re}(C_V^* C_V' - C_S^* C_S') + 2|M_{GT}|^2 \operatorname{Re}(C_A^* C_A' - C_T^* C_T'),$$

$$G = \frac{|M_{GT}|^2}{J(J+1)} \operatorname{Re}(C_A^* C_A' - C_T^* C_T') - \frac{M_F M_{GT}^*}{\sqrt{J(J+1)}} \operatorname{Re}(C_V^* C_A' + C_A^* C_V' - C_S^* C_T' - C_T^* C_S'),$$

$$H = \frac{|M_{GT}|^2}{J(J+1)} (|C_A|^2 + |C_A'|^2 - |C_T|^2 - |C_T'|^2) - \frac{2M_F M_{GT}^*}{\sqrt{J(J+1)}} \operatorname{Re}(C_V^* C_A + C_V'^* C_A' - C_S^* C_T - C_S'^* C_T'),$$

$$W_{F,F_z} \sim \xi - \frac{H}{4} + \frac{(2J+1)G + 2(F-J)(E-G)}{2J+1} F_z \cos\theta + \frac{(F-J)H}{2J+1} \{F(F+1) - F_z^2 + [3F_z^2 - F(F+1)]\cos^2\theta\},$$

$$\xi\beta = [{}^4/_3 (E + 2G) J(J+1) + E] / (2J+1)^2.$$

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A QUALITATIVE INTERPRETATION OF THE MEAN ELECTRON EXCITATION ENERGY IN ATOMIC COLLISIONS

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The transformation of the kinetic energy of the relative motion of colliding particles into the excitation energy of electrons is interpreted as resulting from their deceleration caused by an electron exchange. The motion of electrons in the region of overlapping shells of the colliding particles is considered quasi-classically. It is assumed that, when the electron moves out of the potential field of one of the atoms into that of another, it transfers from the first atom to the second a momentum which, on the average, is equal to the product of the relative velocity of the atoms and the mass of the electrons.

INTRODUCTION

THE majority of experiments on atomic collisions refer to a range of relative velocities of the atoms ($5 \times 10^6 - 10^8$ cm/sec) and of the impact parameters such that one of the excitation theories is not applicable in the first approximation. Thus, for example, it has been shown[1,2] that in collisions between the A atom and A^+ ion with energy equal to 75 kev (velocity equal to 6×10^7 cm/sec) and impact parameter $< 10^{-8}$ cm, the probability of secondary ionization is larger than the probability of elastic collision. However, even the first approximation of the theory of quasi-adiabatic perturbations which could be acceptable for the slowest velocities of the relative motion of atoms ($< 10^7$ cm/sec) turns out to be very complicated. It is very difficult to find the wave functions and the energy levels at all distances between the nuclei. The dependence of the terms on the distance between the nuclei is obviously very complicated, with a large amount of intersections in points where transitions from one term to another occur most frequently. Finally, the very approach of the atoms occurs for already excited states of the produced quasi-molecule, apart from a few exceptional cases. These are the cases where the orbital moments of both atoms equal zero and the sum of their spins is smaller than unity. (The energy level corresponding to separate atoms splits at their approaching each other into many levels, depending on the value of the projection of the atomic moments on the line joining the nuclei and the sum of their spins). The distances between excited levels are very small, and the adiabatic conditions are not fulfilled at comparatively low velocities.

The present paper represents an attempt to estimate at least the average energy of excitation of the electronic shells of colliding atoms using the ideas of classical mechanics. Unfortunately, this could not be sufficiently well based on theory but, from the experimental point of view, the procedure yields reasonable results. The calculation is applicable for distances between the neighboring energy levels of the system of colliding atoms which are small compared to the calculated mean energy of excitation of the electrons. This fact already negates with a high degree of probability, the application of the first approximation of any perturbation theory, since the distances of the first excited level from the ground state are of the same order of magnitude as the ionization energy.

The excitation energy of atomic shells is assumed to be distributed among all electrons and to be lost on ionization of the atoms at the end of the collision process or soon after it. Only that part of energy which is not sufficient for additional ionization can be emitted or be contained in a metastable state. This is connected with the fact, that the time of emission (of the order of $10^{-1} - 10^{-9}$ sec) is much longer than the time for the Auger effect ($10^{-15} - 10^{-14}$ sec) which, in turn, is somewhat longer than the time of collision.

If the average energy of excitation as a function of the relative velocity and of the impact parameter is known, then it is possible to estimate the effective cross section of various inelastic processes using, in addition, statistical considerations. Such an estimate has been carried out by Russel and Thomas.[3]

1. MUTUAL SLOWING DOWN OF ATOMIC NUCLEI DUE TO MOMENTUM TRANSFER BY ELECTRONS

For a sufficiently large number of electrons in each of the colliding atoms, the wave functions of the latter correspond to a high quantum state since they should be antisymmetric with respect to the permutation of electrons. This is also true as far as the wave function of the quasi-molecule constructed out of the atoms in the process of their collision is concerned. For large distances between the nuclei, $R = |\mathbf{R}_a - \mathbf{R}_b|$ the wave function is of the form

$$\Psi = \psi_a(\cdot\cdot\mathbf{r}_a\cdot\cdot)\exp\left(\frac{i}{\hbar}m\dot{\mathbf{R}}_a\Sigma\mathbf{r}_a\right)\psi_b(\cdot\cdot\mathbf{r}_b\cdot\cdot)\exp\left(\frac{i}{\hbar}m\dot{\mathbf{R}}_b\Sigma\mathbf{r}_b\right)$$
$$\times\exp\left[-\frac{i}{\hbar}\left(E + N_a\frac{m\dot{\mathbf{R}}_a^2}{2} + N_b\frac{m\dot{\mathbf{R}}_b^2}{2}\right)\right], \qquad (1)$$

where $\psi_a(\cdot\cdot\mathbf{r}_a\cdot\cdot)$ (N_a electrons) and $\psi_b(\cdot\cdot\mathbf{r}_b\cdot\cdot)$ (N_b electrons) are the wave functions of atoms for stationary nuclei, $\mathbf{r}_{a,b}$ are the coordinates of electrons and $\dot{\mathbf{R}}_{a,b}$ are the velocities of the atomic nuclei. Evidently, it is necessary to carry out a permutation of electrons belonging to various atoms.

The exponential factors indicate the fact that the average velocity of each electron is identical with the velocity of the atom to which it belongs, and that the kinetic energy due to the motion of the atom as a whole must be added to the energy of electrons in stationary nuclei. The relation between the mean electron velocity and their position with respect to the nuclei is in some manner conserved in the collision process. In that main part of the distribution of the electron concentration in atoms or molecules, the energy of the interaction of electron pairs is equal to $\sim e^2n^{1/3}$, (where e is the charge and n the concentration of electrons) and is small with their kinetic energy $\sim \hbar^2n^{2/3}/2m$. The motion of electrons takes place to a first approximation, in the field of electrons and nuclei determined from the Thomas-Fermi model.

The Thomas-Fermi potential has on the line connecting the atomic nuclei a saddle-type minimum, through which we shall construct a surface S perpendicular to the equipotential surfaces. (For different charges of the atomic nuclei, this is an infinite surface perpendicular to the line connecting the nuclei, and dividing it into two). From the Gauss theorem we know that if the quasi-molecule is neutral on the whole, then the total charge of electrons which on the average are present on one side of this surface is equal to the charge of the corresponding nucleus.

For a motion in the direction of the corresponding nucleus along this surface, or any equipotential surface, the potential increases faster than the inverse distance to this nucleus. Trajectories of electrons moving in such fields according to the laws of classical mechanics would in general be loops or even twisting and untwisting spirals.

It is therefore natural to assume that the surface S divides the regions of the action of the potentials of the first and second atom. Passing through S, the electrons strongly interact with the field of the corresponding atom, losing their initial momentum and, on the average, assuming a momentum corresponding to the velocity of the atom. The electron flux density through the element of area dS in one direction is nv/4, where v is the mean value of the absolute velocity. (We assume that the velocity distribution of electrons is spherically symmetric). The total momentum transfer, i.e., the force acting on the corresponding atom, will be given by the formula

$$\mathbf{F} = \pm m(\dot{\mathbf{R}}_a - \dot{\mathbf{R}}_b)\int_S \frac{nv}{4}\,dS. \qquad (2)$$

A similar force with opposite sign acts on the atom. The work done by these forces moving the atoms for a distance $d\mathbf{R}_a$ and $d\mathbf{R}_b$ is equal to $\sim -m(\dot{\mathbf{R}}_a - \dot{\mathbf{R}}_b)\,d(\mathbf{R}_a - \mathbf{R}_b)$ or, denoting $d\mathbf{R}_a - d\mathbf{R}_b = d\mathbf{R}$, we obtain for the total work of slowing down the atoms (in other words, the electron excitation energy)

$$\mathscr{E} = m\int\left(\int_S \frac{nv}{4}\,dS\right)\dot{\mathbf{R}}\,d\mathbf{R}. \qquad (3)$$

Assuming that the interaction does not very strongly affect the expression given by the Thomas-Fermi model, $v = \frac{3}{4}(3\pi^2)^{1/2}\hbar n^{1/3}/m$,

$$\mathscr{E} = \frac{3}{16}(3\pi^2)^{1/3}\hbar\int\left(\int_S n^{4/3}dS\right)\dot{\mathbf{R}}\,d\mathbf{R}, \qquad (4)$$

or, expressing n in terms of the potential φ using $n = 2^{3/2}(me\varphi)^{3/2}/3\pi^2\hbar^3$, and substituting into Eq. (4), we have

$$\mathscr{E} = \frac{m^2e^2}{4\pi^2\hbar^3}\int\left(\int_S \varphi^2 dS\right)\dot{\mathbf{R}}\,d\mathbf{R}. \qquad (5)$$

In deriving Eq. (2), we did not take into account the transfer of momentum due to the exchange of electrons (or to the collision of electrons belonging to various atoms) since the corresponding force of repulsion between the atoms is also present for stationary atoms and is a part of the conservative forces.

In addition, it was assumed $|\dot{\mathbf{R}}| \ll v$. In the opposite case, the momentum transport would be much smaller.

2. ESTIMATE OF THE MEAN EXCITATION ENERGY OF ELECTRONS ACCORDING TO EQ. (5)

Taking into account the qualitative character of Eq. (5), several other simplifications have been made in the following calculation.

1. If the discussion is limited to small-angle scattering, the motion of the nuclei is assumed to be rectilinear and uniform (the loss of kinetic energy is small compared to its initial value). We have then $\dot{\mathbf{R}}d\mathbf{R} = u\,dx$, where $u = |\dot{\mathbf{R}}|$, $dx = \dot{\mathbf{R}}d\mathbf{R}/|\dot{\mathbf{R}}|$, and

$$\int\left(\int_S \varphi^2 dS\right)\mathbf{R}d\mathbf{R} = u\int_{-\infty}^{\infty}\left(\int_S \varphi^2 dS\right)dx. \qquad (6)$$

From Eqs. (5) and (6), we obtain, taking φ and all linear dimensions in atomic units (for a potential e/a_0, $a_0 = \hbar^2/me^2$),

$$\mathscr{E} = \frac{\hbar u}{4\pi^2 a_0}\int_{-\infty}^{\infty}\left(\int_S \varphi^2 dS\right)dx. \qquad (7)$$

2. If the nuclear charges of the atoms differ by not more than approximately four times, then, with a good accuracy, we can assume the surface S to be a plane perpendicular to the line connecting the nuclei and dividing it in two. For the potential φ at this plane, we assume

$$\varphi = \frac{Z_a + Z_b}{r}\chi\left(1.13\,[Z_a + Z_b]^{1/3} r\right), \qquad (8)$$

where r is the distance from the point of the plane to one of the nuclei. Furthermore, since $dS = 2\pi\rho\,d\rho$, and $r^2 = (R/2)^2 + \rho^2$, we introduce a new variable ξ

$$\xi = 1.13\,(Z_a + Z_b)^{1/3}\sqrt{(x/2)^2 + \rho^2},$$
$$\xi_0 = 1.13\,(Z_a + Z_b)^{1/3} R_0/2, \qquad (9)$$

where R_0 is the minimum distance between the nuclei ($R^2 = R_0^2 + x^2$), and transform Eq. (7) with potential φ determined by Eq. (8) into

$$\mathscr{E} = \frac{1.77}{\pi}(Z_a + Z_b)^{5/3}\frac{\hbar u}{a_0}\int_0^{\infty}\chi^2\left(V\overline{\xi_0^2 + \xi^2}\right)\frac{\xi^2 d\xi}{\xi_0^2 + \xi^2}. \qquad (10)$$

The integral in Eq. (10) is evaluated numerically and can be well approximated by the function

$$f = 0.61/(1 + 0.285\,\xi_0)^5 \qquad (11)$$

up to the value of $\xi_0 = 6$. For large values of ξ_0, Eq. (11) gives a slightly lower value than the integral in Eq. (10).

We have, therefore

$$\mathscr{E} = \frac{0.35\,(Z_a + Z_b)^{5/3}\,\hbar u/a_0}{[1 + 0.16\,(Z_a + Z_b)^{1/3} R_0/a_0]^5} \qquad (12)$$

or, in electron-volts

$$\mathscr{E} = \frac{(Z_a + Z_b)^{5/3}\,4.3\cdot 10^{-8} u}{[1 + 3.1\,(Z_a + Z_b)^{1/3} 10^7 R_0]^5} \qquad (13)$$

where u is expressed in cm/sec and R_0 in centimeters.

3. COMPARISON WITH EXPERIMENT

Direct measurements of the average energy loss are known only for collisions of the ions of A^+ and Ne^+ at an energy of 75 kev with the atoms of A (experiments of Abrosimov and Fedorenko[2] on the scattering of argon target atoms at angles in the range of 84 to 78°). Theoretical values of the impact parameter R_0 necessary for calculations are taken from reference 4. We obtain the following table for the values of $\mathscr{E}$:

TABLE I

	75 kev $A^+ + A$, $u = 6\times10^7$ cm/sec				75 kev $Ne^+ + A$, $u = 8.5\times10^7$ cm/sec			
ϑ, deg	84	82	80	78	84	82	80	78
$10^9\cdot R_0$	2.3	1.9	1.7	1.54	2.2	1.8	1.54	1.48
$\mathscr{E}_{theor}$, ev	340	400	440	510	350	410	450	510
$\mathscr{E}_{exp}$, ev	380	640	890	990	490	560	750	940

The largest value of the mean average energy loss, as far as it can be judged from an extrapolation of the experimental data, is ~ 1500 ev for collision of $A^+ + A$ and $Ne^+ + A$, while the theoretical value for both cases is ~ 1000 ev.

Apart of this direct comparison with the experiment (other data for direct comparison are not yet available), we can make an indirect comparison.

By solving Eq. (13) with respect to R_0, we can obtain the cross section πR_0^2 for an average electron excitation energy exceeding $\mathscr{E}_0$. Inelastic processes that require an energy greater than $\mathscr{E}_0$ will have in this case a lesser probability. On the other hand, if the excitation energy is greater than the ionization energy, ionization occurs with a high probability since the time of emission is large compared with the time of the Auger effect which is comparable with the collision time.

Assuming, therefore, that $\mathscr{E}_0$ is equal to the smaller ionization energy of the two colliding atoms, one can approximately estimate the cross section for removing electrons as a function of the relative velocity of atoms and the number of electrons in then:

$$\pi R_0^2 = 3.3\cdot 10^{-15}(Z_a + Z_b)^{-2/3} \times [((Z_a + Z_b)^{5/3} 4.3\cdot 10^{-8} u/\mathscr{E}_0)^{1/3} - 1]^2 \text{ cm}^2 \qquad (14)$$

($\mathscr{E}_0$ in ev). Introducing the characteristic velocity and cross section for each pair of colliding atoms

$$u_0 = [23\cdot 10^6 \mathscr{E}_0/(Z_a + Z_b)^{5/3}] \text{ cm/sec},$$

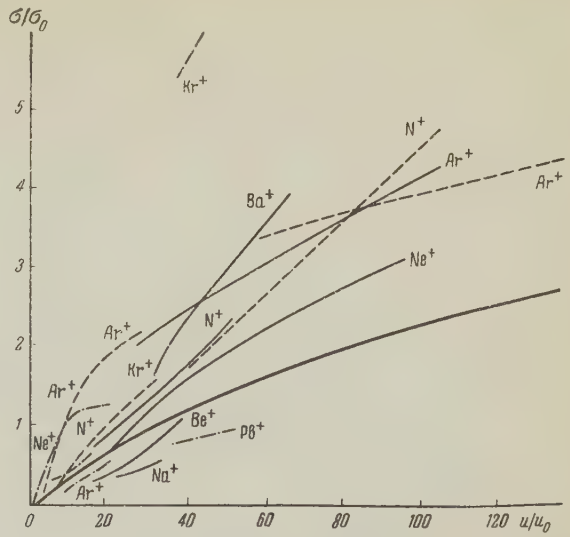
$$\sigma_0 = [33\cdot 10^{-16}/(Z_a + Z_b)^{2/3}] \text{cm}^2, \qquad (15)$$

the formula (14) can be written in the form

$$\sigma/\sigma_0 = [(u/u_0)^{1/3} - 1]^2, \qquad (16)$$

which gives a universal dependence for the cross section for the removal of electrons for any colliding pair as a function of their relative velocity.

Such a comparison with the available experimental data was carried out by N. V. Fedorenko. This comparison is given in the figure and in Table II, with his kind permission.



Dependence of the cross section of electron removal in colliding pair of atoms on the relative velocity in A (solid curve), Kr (dotted curve), and Ne (dot-dash curve). Solid, heavy line—theoretical.

4. DISCUSSION OF RESULTS

Comparison with the experiment, both direct and indirect, shows that experimental and theoretical data differ essentially by not more than a factor of two (with a few exceptions in the indirect comparison). It should be noted that, in the indirect comparison with the experiment, we studied eight different pairs of colliding atoms which are greatly different both in the total charge (from $N^+ + Ne$, $Z_a+Z_b = 16$ to $Pb^+ + Ne$, $Z_a + Z_b = 92$; $Ba^+ + A$, $Z_a + Z_b = 73$), and with respect to the ratio of their charge. The range of the relative velocities varies by a factor of ~ 30. From a comparison of $Ba^+ + A$ and $Pb^+ + Ne$, it can also be seen that the deviation of experimental data from theory is not due to an incorrect dependence on the sum of charge or on their ratio (the deviations are in both directions), but rather to individual peculiarities. The latter cannot be taken into account in the theory if the motion of electrons is considered according to the statistical model of Thomas and Fermi.

As to the direct comparison with the experiment, the theoretical result shows a weaker dependence of the mean energy of excitation on the impact parameter than that obtained experimentally. Obviously, this deviation is within the limits of accuracy of the calculation since, first, there is no solution for the Thomas-Fermi equation for two nuclei and, second, the impact parameter may be uncertain by as much as 10 or 20%. The theory given above cannot be expected to show a better agreement with the experiment, especially if one takes into account the roughness of the calculation carried out within its framework and the indirect character of the comparison with the experiment. We would still like to mention that the deviation of some experimental points in the direction of increasing cross section

TABLE II

Pair	$\sigma_0\cdot 10^{16}$, cm²	$u_0\cdot 10^{-6}$, cm/sec	Velocity range 10^6 cm/sec	Reference	Pair	$\sigma_0\cdot 10^{-16}$ cm²	$u_0\cdot 10^{-6}$, cm/sec	Velocity range 10^6 cm/sec	Reference
$N^+ + Ne$	4.94	4.48	25÷65	[5]	$Ar^+ + Ar$	—	—	24÷95	[6]
$Ne^+ + Ne$	4.44	3.42	3.42÷14.0	[7]	$Ba^+ + Ar$	1.86	0.281	8.5÷18	[5]
$Ne^+ + Ne$	—	—	22.5÷63.0	[7]	$Be^+ + Ar$	4.17	2.11	32÷80	[5]
$Ar^+ + Ne$	3.54	1.95	16.5÷39.0	[5]	$N^+ + Kr$	2.70	0.642	25÷66	[5]
$Pb^+ + Ne$	1.61	0.269	9.5÷13.7	[5]	$Ar^+ + Kr$	2.29	0.423	1.1÷11.4	[7]
$N^+ + Ar$	3.82	1.68	26÷65	[5]	$Ar^+ + Kr$	—	—	24÷60	[6]
$Na^+ + Ar$	3.46	1.33	28÷43	[5]	$Kr^+ + Kr$	1.89	0.262	1.8÷8.0	[7]
$Ne^+ + Ar$	3.54	1.41	30÷132	[6]	$Kr^+ + Kr$	—	—	10÷12	[7]
$Ar^+ + Ar$	3.00	0.91	2.2÷10	[7]					

can be explained. First, when the velocity of relative motion is much greater than the minimal necessary for the ionization, the cross section for double or even triple ionization has the same order of magnitude as for single ionization. Therefore, two or three electrons respectively are often captured in the collision, thus increasing the cross section. Secondly, as mentioned in the beginning of this article, in collisions, the quasi-molecule is produced adiabatically already in the excited state, and if the removal of an electron occurs at a comparatively small distance between the atoms, then the energy of removal may be two to three times smaller than the energy of ionization for far away atoms.

Correspondingly, the ratio u/u_0 will be three times larger. Taking into account these two facts, the discrepancy with experiment will be much smaller.

In any case, formula (16) gives the cross section for ionization for any pair of colliding atoms and in greater range of relative velocities with an accuracy to a factor of two.

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AZIMUTHAL SYMMETRIES IN CASCADE REACTIONS AND PARITY CONSERVATION

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Some angular azimuthal symmetry relations are obtained for cascades of reactions, of the type of triple scattering of protons; these relations follow from parity conservation in the reactions of the cascade. It is pointed out that experimental confirmation of the simplest of these relations, which is well known — the symmetry of the particles from the second scattering with respect to the plane of the first scattering — is not an exhaustive check on parity conservation. Experiments suggested here are a more thorough check of this law, and in some cases amount to a complete proof.

In the present paper the following azimuthal symmetry relations are obtained.

1. Suppose there is triple scattering by unpolarized targets (the incident beam for the first reaction is also unpolarized). If "parity is conserved"* in all the reactions, then the number of particles that undergo second scattering in the direction (ϑ, φ) and third scattering in the direction (ϑ', φ') is equal to the number of particles undergoing second scattering at the angles $(\vartheta, -\varphi)$ and further scattering (by a suitably placed target) at the angles $(\vartheta', -\varphi')$:

$$\sigma_{\vartheta,\varphi}(\vartheta',\ \varphi') = \sigma_{\vartheta,-\varphi}(\vartheta',\ -\varphi'). \tag{1}$$

The angles ϑ, φ are measured in the following coordinate system: the z axis is along the direction of the first scattering, and the y axis is perpendicular to the plane of the first reaction. For the angles ϑ', φ' the z axis is parallel to the direction (ϑ, φ), and the y axis is perpendicular to the plane of the second reaction (for details see Sec. 3).

The symmetry (1) is valid for any cascade of the type

$$a + b \to c + d,\quad (\vartheta_0,\ \varphi_0),\quad c + e \to f + g,\ (\vartheta,\ \varphi),\quad f + h \to i + j,\ (\vartheta',\ \varphi'), \tag{2}$$

if the particles a and the targets b, e, h are unpolarized. a, b, c, etc. can be nuclei or "elementary" particles (including γ-ray quanta) with arbitrary spins.

*We shall say for brevity that "parity is conserved" if: 1) in the real three-dimensional space there is no distinction between a right-handed and a left-handed screw; 2) all the particles taking part in the reactions have definite parities. "Parity is not conserved" if any one of these postulates does not hold.

2. If "parity is conserved" in the reactions of the cascade

$$a + b \to a + b,\qquad \begin{matrix} a + e_1 \to a + e_1,\quad (\vartheta_1,\ \varphi_1) \\ b + e_2 \to b + e_2,\quad (\vartheta_2,\ \varphi_2) \end{matrix} \tag{3}$$

with unpolarized beam a and targets b, e_1, e_2, the number of coincidences for counters of the twice-scattered particles a and b placed in the directions (ϑ_1, φ_1) and (ϑ_2, φ_2) must equal the number of coincidences with the counters placed in the directions $(\vartheta_1, -\varphi_1)$ and $(\vartheta_2, -\varphi_2)$:

$$\sigma(\vartheta_1,\ \varphi_1;\ \vartheta_2,\ \varphi_2) = \sigma(\vartheta_1, -\varphi_1;\ \vartheta_2 - \varphi_2). \tag{4}$$

Some of the reactions in the sequences (2) and (3) can be replaced by particle-decay reactions of the type $a \to c + d$. For example, Eq. (1) holds for the cascade $K^- + n \to \Xi^- + K^+$, $\Xi^- \to \Lambda + \pi$, $\Lambda \to p + \pi$, and Eq. (4) holds for the cascade $\pi^- + p \to \Sigma^- + K^+$, $\Sigma^- \to n + \pi$, $K \to \pi + \pi$.

The proposed experiments essentially serve to complete the set of experiments required to reconstruct the transition matrix (S matrix) of the reactions; this set is usually laid out on the assumption that parity is conserved. Of course, if any of these symmetries (or any already known) is violated, then parity is not conserved.

1. GENERAL FORMULAS

Formulas are known in the literature that express, in terms of the elements of the reaction matrix $R = S - 1$, the angular distribution of the products of the reaction of a polarized beam and polarized target, and also the polarization vector of these products (and if their spins are larger than $\frac{1}{2}$, also the polarization tensors).

Introducing instead of rectangular components of the spin vector the cyclic components,

$$\sigma_{-1} = (\sigma_x + i\sigma_y)/\sqrt{2}, \qquad \sigma_0 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_{+1} = -(\sigma_x - i\sigma_y)/\sqrt{2},$$

we can write these formulas for the case of scattering of a particle with spin $\frac{1}{2}$ by a spinless particle (cf. references 1 and 2) as follows:

$$I = \frac{1}{2}\left\{\mathrm{Sp}\, RR^+ + \sum_{\tau=-1}^{+1} P_\tau^{in}\, \mathrm{Sp}\, R\tilde{\sigma}_\tau R^+\right\} \tag{5}$$

is the angular distribution for polarized incident beam; and

$$P'_{\tau'} I = \frac{1}{2}\left\{\mathrm{Sp}\, \sigma_{\tau'} RR^+ + \sum_{\tau=-1}^{+1} P_\tau^{in}\, \mathrm{Sp}\, \sigma_{\tau'} R\tilde{\sigma}_\tau R^+\right\} \tag{6}$$

gives the cyclic components of the polarization vector of the scattered particles, for polarized incident beam ($\sim$ denotes transposition).

Let us write Eqs. (5) and (6) compactly in the form

$$\rho'(\vartheta,\ \varphi;\ q'\tau') = \sum_{q=0}^{1} \sum_{\tau=-q}^{+q} (q'\tau' | W(\vartheta,\ \varphi) | q\tau)\, \rho(q,\ \tau), \tag{7}$$

where $\rho(0,\ 0) = 1$ (one particle in the target and unit flux density of incident particles); $\rho(1,\ \tau) \equiv P_\tau^{in}$ are the cyclic components of the polarization vector of the beam; $\rho'(\vartheta, \varphi;\ 00) \equiv I(\vartheta,\ \varphi)$ gives the angular distribution; $\rho'(\vartheta, \varphi;\ 1\tau') \equiv P'_{\tau'}(\vartheta, \varphi) \times I(\vartheta,\ \varphi)$; and

$$(1\tau' | W(\vartheta,\ \varphi) | 1\tau) \equiv {}^1/_2\, \mathrm{Sp}\, \sigma_{\tau'} R(\vartheta,\ \varphi)\, \tilde{\sigma}_\tau R^+(\vartheta,\ \varphi). \tag{8}$$

For any reaction $a + b \to c + d$ (the spins are arbitrary) a formula of this kind also holds (cf. reference 3, and also references 4 – 6):

$$\rho'(\vartheta,\ \varphi;\ q_c\tau_c q_d\tau_d) = \sum_{q_a\tau_a q_b\tau_b} (q_c\tau_c q_d\tau_d | W(\vartheta,\ \varphi) | q_a\tau_a q_b\tau_b)\, \rho(q_a\tau_a q_b\tau_b); \tag{9}$$

$$\begin{aligned}(q_c\tau_c q_d\tau_d | W(\vartheta,\ \varphi) | q_a\tau_a q_b\tau_b) &= [(2i_c+1)(2i_d+1)]^{1/2} [(2i_a+1)(2i_b+1)]^{-1/2} \\ &\times \sum_{m_c\, m'_c, m_d, m'_d} (-1)^{i_c - m'_c} (i_c i_c m_c - m'_c | q_c\tau_c) (-1)^{i_d - m'_d} \\ &\times (i_d i_d m_d - m'_d | q_d\tau_d) \sum_{m_a, m_a', m_b, m_b'} (m_c m_d | R(\vartheta,\ \varphi) | m_a m_b) \\ &\times (m_c' m_d' | R(\vartheta,\ \varphi) | m_a' m_b')^* (-1)^{i_a - m'_a} \\ &\times (i_a i_a m_a - m_a' | q_a\tau_a) (-1)^{i_b - m'_b} (i_b i_b m_b - m'_b | q_b\tau_b).\end{aligned} \tag{10}$$

Letters i denote the spins of the particles, m their z components, q the rank of polarization tensors (for definition see reference 4). In this general case the Clebsch-Gordan coefficients $(\mathrm{iim} - \mathrm{m}' | \mathrm{q}\tau)$ appear instead of the matrices σ_τ of Eqs. (5), (6), and (8).

There is an analogous formula for the decay reaction $a \to c + d$ (cf. reference 4):

$$\rho'(\vartheta,\ \varphi;\ q_c\tau_c q_d\tau_d) = \sum_{q_a\nu_a} (q_c\tau_c q_d\tau_d | W(\vartheta,\ \varphi) | q_a\nu_a)\, \rho(q_a\nu_a).$$

If the total angular momentum is conserved, then for the elements of the transition matrix of the reaction $a + b \to c + d$ in the representation of momenta and spin components we have the well known expression [cf., e.g., reference 9, Eq. (4)]:

$$\begin{aligned}(\mathbf{p}_c n_c n_d | R | \mathbf{p}_a n_a n_b) &= N \sum Y_{l'\mu'}(\vartheta_c, \varphi_c)\, C^{s'n'}_{i_c n_c i_d m_d} C^{JM}_{l'\mu' s'n'} \\ &\times (s'l' | R^J | sl)\, C^{JM}_{l\mu sn} C^{sn}_{i_a n_a i_b n_b}\, Y_{l\mu}(\vartheta_a,\ \varphi_a),\end{aligned} \tag{11}$$

N is a normalizing factor. The sum is taken over all indices that occur twice. In this formula all the quantities are referred to a single coordinate system zyx.

In Eq. (10) and everywhere hereafter it is assumed that the spin components m_a and m_b of particles a and b are referred to a reference system A with the axis $z_a \parallel \mathbf{p}_a$. The y axis is chosen, for example, parallel to the polarization vector of a or b. The components m_c and m_d, on the other hand, are referred to the system C: $z_c \parallel \mathbf{p}_c$, $y_c \parallel [\mathbf{p}_a \mathbf{p}_c]$.[4] Going over from the old z axis of quantization to the new axes z_a and z_c, we get from Eq. (11) the following expression for the elements of R in the representation of these spin components:*

$$\begin{aligned}(m_c m_d | R(\vartheta,\ \varphi) | m_a m_b) &= \frac{N}{4\pi} \sum_{s', l', J, l, s} C^{s' m_c + m_d}_{i_c m_c i_d m_d} C^{J m_c + m_d}_{s' m_c + m_d\, l' 0} \\ &\times [(2l'+1)(2l+1)]^{1/2} (s'l' | R^J | sl)\, C^{J m_a + m_b}_{s m_a + m_b\, l 0} C^{s m_a + m_b}_{i_a m_a i_b m_b} \\ &\times D^J_{m_c + m_d, m_a + m_b}(-\pi,\ \vartheta,\ \pi - \varphi).\end{aligned} \tag{12}$$

The function D is defined in reference 4. It follows from Eq. (12) such elements of R do not depend on $\mathbf{p}_c$ and $\mathbf{p}_a$ separately, but on the Euler angles $\{-\pi,\ \vartheta,\ \pi - \varphi\}$ of the rotation that makes the axes A coincide with the axes C; ϑ and φ are the spherical angles of the vector $\mathbf{p}_c$ relative to the axes A (cf. reference 4, Sec. 2). In addition, the dependence on φ is known:

$$\begin{aligned}&(m_c m_d | R(\vartheta,\ \varphi) | m_a m_b) \\ &= (m_c m_d | R(\vartheta,\ 0) | m_a m_b)\, e^{i(m_a + m_b)\varphi}.\end{aligned} \tag{13}$$

2. THE PARITY SELECTION RULE

In Eq. (12) let us make the substitution

$$\begin{aligned}D^J_{m_c + m_d, m_a + m_b}(-\pi,\ \vartheta,\ \pi - \varphi) &= (-1)^{m_a + m_b - m_c - m_d} \\ &\times D^J_{-m_c - m_d, -m_a - m_b}(-\pi,\ \vartheta,\ \pi + \varphi),\end{aligned}$$

*Analogous calculations are carried out in Sec. 2 of reference 4 [derivation of Eq. (2.7) from Eq. (2.4)].

use the property

$$C^{c\gamma}_{a\alpha b\beta} = (-1)^{\pm c \mp a \mp b} C^{c-\gamma}_{a-\alpha\, b-\beta}$$

of the Clebsch-Gordan coefficients, and also use the well known expression of the law of conservation of parity (cf. footnote*): $\pi_c\pi_d(-1)^{l'} = \pi_a\pi_b(-1)^{l}$. We finally get

$$(m_c m_d | R(\vartheta, \varphi) | m_a m_b) = \pi_c^* \pi_d^* \pi_a \pi_b (-1)^{i_c+i_d+i_a+i_b-m_c-m_d-m_a-m_b} \times (-m_c - m_d | R(\vartheta, -\varphi) | -m_a - m_b). \quad (14)$$

By the same method as in Sec. 1 of reference 5 we get the corresponding relation for the coefficients W:

$$(q_c\tau_c q_d\tau_d | W(\vartheta, \varphi) | q_a\tau_a q_b\tau_b) = (-1)^{q_c+q_d+q_a+q_b+\tau_c+\tau_d+\tau_a+\tau_b} \times (q_c - \tau_c q_d - \tau_d | W(\vartheta, -\varphi) | q_a - \tau_a q_b - \tau_b). \quad (15)$$

The components are referred to the directions of the respective relative momenta $\mathbf{p}_c$ and $\mathbf{p}_a$.

This "selection rule" was first obtained by Chou Kuang-Chao[7] in a different formulation. L. G. Zastavenko has pointed out that it is completely equivalent to the rule "$\nu_c+\nu_d+\nu_a+\nu_b$ even" obtained in reference 4. For decay reactions $a \to c + d$ we have an entirely analogous relation; we have only to remove the labels q_b and τ_b in Eq. (15).

Since the dependence of W on φ is known [cf. reference 5, Eq. (6)] both sides of Eq. (15) can be divided by $\exp\{i(\tau_a+\tau_b)\varphi\}$, i.e., without any loss of generality we can set $\varphi = 0$ in Eq. (15) [and Eq. (14), see Eq. (13)].

Taking the complex conjugates of both members of Eq. (10), and noting also that $(iim - m' | q\tau) = (iim' - m | q - \tau)$ and $(-1)^{-m'} = (-1)^{\tau-m}$, we get

$$(q_c\tau_c q_d\tau_d | W(\vartheta, \varphi) | q_a\tau_a q_b\tau_b)^* = (-1)^{\tau_c+\tau_d+\tau_a+\tau_b} \times (q_c - \tau_c q_d - \tau_d | W(\vartheta, \varphi) | q_a - \tau_a\, q_b - \tau_b). \quad (16)$$

From Eqs. (16) and (15) there follows one more formulation of the parity "selection rule"

$$(q_c\tau_c q_d\tau_d | W(\vartheta,0) | q_a\tau_a q_b\tau_b) = (-1)^{q_c+q_d+q_a+q_b} (q_c\tau_c q_d\tau_d | W(\vartheta, 0) | q_a\tau_a q_b\tau_b)^*, \quad (17)$$

i.e., the coefficients $W(\vartheta, 0)$ are purely real if $q_c + q_d + q_a + q_b$ is even, and purely imaginary if this sum is odd.

3. DERIVATION OF THE AZIMUTHAL SYMMETRIES

Let us express the above selection rule in terms of properties of the angular distributions that are directly observable in experiments.

If a and b in the cascade (2) are unpolarized, then

$$\rho_{ab}(q_a\tau_a q_b\tau_b) = 1 \cdot \delta_{q_a,0}\delta_{q_b,0}$$

(if the flux density of the incident particles is unity and there is one target particle). The polarization tensors of particle c (we do not concern ourselves with particle d) are then independent of the azimuthal angle φ_0 and have the property (cf. reference 4)

$$\rho_0(\vartheta_0; q_c\tau_c 00) = (q_c\tau_c 00 | W_0(\vartheta_0, 0) | 0000) = (-1)^{q_c+\tau_c}\rho_0(\vartheta_0; q_c - \tau_c 00). \quad (18)$$

In order to find the polarization tensors of the product f of the second reaction of the cascade (2), we have to substitute in Eq. (9) the polarization tensors $\tilde{\rho}_0$ of particle c, referred to the axes $\tilde{C}$ belonging to the initial state of the second reaction (stationary relative to its center of mass). But the tensors (18) are referred to C_0, the axes belonging to the first reaction (stationary relative to the center of mass of the first reaction). To bring the axes C_0 into coincidence with the axes $\tilde{C}$ it is enough just to rotate the axis z_{c0}, parallel to the momentum $\mathbf{p}_c$, around y_{c0} by such an angle α that it becomes parallel to the direction $\mathbf{p}_c^l$ of this momentum in the laboratory system (we note that this direction is the same as its direction in the center-of-mass system of the second reaction, since the target e is at rest).

Therefore

$$\tilde{\rho}_0(\vartheta_0; q_c\tilde{\tau}_c) = \sum_{\tau_c} D^{q_c}_{\tilde{\tau}_c,\tau_c}(0, \alpha, 0)\rho_0(\vartheta_0; q_c\tau_c). \quad (19)$$

We now insert in Eq. (19) instead of $\rho_0(\vartheta_0; q_c\tau_c)$ the quantities $(-1)^{q_c+\tau_c}\rho_0(\vartheta_0; q_c - \tau_c)$ which are equal to the former quantities by Eq. (18), change the sign of the summation index, $\tau_c \to -\tau_c$, and use the relation

$$D^{q}_{\tilde{\tau},-\tau}(0, \alpha, 0) = (-1)^{-\tau-\tilde{\tau}} D^{q}_{-\tilde{\tau},\tau}(0, \alpha, 0).$$

We find in this way that $\tilde{\rho}_0$ has the same property (18) as ρ_0. And in general rotations of the sets of axes belonging to the products of a reaction around the perpendicular to the plane of the reaction do not change the formulation (15) of the parity selection rule.

The relativistic spin rotation[9] is also applied around the perpendicular to the plane of the reaction. We can assume that α already includes the angle Ω of this rotation, and therefore the azimuthal symmetries obtained in what follows will be relativistic results.

Thus we have for the polarization tensors f

$$\rho(\vartheta,\ \varphi;\ q_f\tau_f 00)$$

$$= \sum_{q_c\tau_c} (q_f\tau_f 00\,|\,W(\vartheta,\ \varphi)|\ q_c\tau_c 00)\,\tilde{\rho}_0(\vartheta_0;\ q_c\tau_c 00)\cdot 1,$$

if the target e is unpolarized (so that $\rho(q_e\tau_e) = 1\cdot\delta_{q_e,0}$). Since τ_c ranges from $-q_c$ to $+q_c$, we can replace the summation index τ_c by $-\tau_c$ (regrouping of the terms of the sum). Then applying Eq. (15) and for brevity not writing out the indices q and τ when they are equal to zero we have

$$\rho(\vartheta,\ \varphi;\ q_f\tau_f) = (-1)^{q_f+\tau_f} \sum_{q_c\tau_c} (q_f - \tau_f\,|\,W(\vartheta,\ -\varphi)|\ q_c\tau_c)$$

$$\times\,\tilde{\rho}_0(\vartheta_0;\ q_c\tau_c) = (-1)^{q_f+\tau_f}\rho(\vartheta,\ -\varphi;\ q_f-\tau_f).$$

Use has been made of the equation $(-1)^{q_c+\tau_c} \times (-1)^{q_c+\tau_c} = 1$. The angles ϑ, φ are measured from the respective axes $\tilde{z}_c \parallel \mathbf{p}_c^l$; $\tilde{y}_c \parallel \mathbf{p}_a \times \mathbf{p}_c^l$.

In particular, for $q_f = 0$ we get $\sigma(\vartheta, \varphi) = (\vartheta, -\varphi)$, i.e., symmetry of the angular distribution of the products of the second reaction with respect to the plane of the first reaction.

For the polarization tensors of the products of the third reaction we get in the same way, if the target h is unpolarized:

$$\rho'_{\vartheta,\varphi}(\vartheta',\ \varphi';\ q_i\tau_i q_f\tau_f) = \sum_{q_f\tau_f} (q_i\tau_i q_f\tau_f\,|\,W'(\vartheta',\ \varphi')|\ q_f\tau_f)$$

$$\times \sum_{q_c\tau_c} (q_f\tau_f\,|\,W(\vartheta,\ \varphi)|\ q_c\tau_c)\,(q_c\tau_c\,|\,W_0(\vartheta_0,\ 0)|)$$

$$= (-1)^{q_i+\tau_i+q_j+\tau_j}\,\rho'_{\vartheta,-\varphi}(\vartheta',\ -\varphi';\ q_i-\tau_i q_j-\tau_j). \quad (20)$$

The indices ϑ, φ on ρ' mean that the particles f incident on h are those that emerged at the angles ϑ, φ with respect to the axes $\tilde{C}$ of the second reaction. The angles ϑ', φ' are measured relative to the axis system $\tilde{F}$: $\tilde{z}_f$ is parallel to $\mathbf{p}_f^l$, the momentum of f in the laboratory system, and $\tilde{y}_f \parallel \mathbf{p}_c^l \times \mathbf{p}_f^l$. In particular, for the angular distribution in the third reaction we get the symmetry (1). The extension to cascades with an arbitrary number of reactions is obvious (all the azimuthal angles φ in the right members of the equations are replaced by $-\varphi$).

In establishing the symmetry (1) we have used out of all the relations (15) only those of the form

$$(q_c\tau_c 00\,|\,W(\vartheta,\ \varphi)|\ q_a\tau_a 00) = (-1)^{q_c+q_a+\tau_c+\tau_a}$$

$$\times\,(q_c-\tau_c 00\,|\,W(\vartheta,\ -\varphi)|\ q_a-\tau_a 00), \quad (21)$$

where q_c, q_a take the values $0, 1, \ldots\ldots 2i_c$ and $0, 1, \ldots\ldots 2i_a$, respectively.

Some of the other relations of Eq. (15) are used in proving the symmetry (4). In our arguments so far we have not been concerned at all with the second products of the reactions. But the common origin of the products of the reaction $a + b \to a + b$ makes the angular distributions of the particles a and b after second scattering correlated (regarding the correlation of the polarizations cf., e.g., reference 8). Namely, let us select among all the particles a after their first scattering only those that emerged together with particles b that have undergone subsequent scattering in the direction ϑ_2, φ_2. The angular distribution $\sigma(\vartheta_1, \varphi_1)$ from the second scattering of this subensemble of particles a will depend on ϑ_2 and φ_2 as parameters. The selection is made by the usual method of coincidences. For this joint angular distribution of the second scatterings we get by using Eq. (15):

$$\sigma(\vartheta_1,\ \varphi_1;\ \vartheta_2,\ \varphi_2)$$

$$= \sum_{q_a\tau_a q_b\tau_b} (|W_1(\vartheta_1,\ \varphi_1)|\ q_a\tau_a)\,(|W_2(\vartheta_2,\ \varphi_2)|\ q_b\tau_b)$$

$$\times\,(q_a\tau_a q_b\tau_b\,|\,W_0(\vartheta_0,\ 0)|) = \sigma(\vartheta_1,\ -\varphi_1;\ \vartheta_2,\ -\varphi_2).$$

By the same method we get for the cascade

$$a + b \to c + d, \quad c + e \to f + g, \ (\vartheta,\ \varphi)$$

$$f + h_1 \to i_1 + j_1, \ (\vartheta_1',\ \varphi_1')$$
$$g + h_2 \to i_2 + j_2, \ (\vartheta_2',\ \varphi_2')$$

with unpolarized a, b, e, h_1, h_2 the result

$$\sigma_{\vartheta,\varphi}(\vartheta_1',\ \varphi_1';\ \vartheta_2',\ \varphi_2')$$

$$= \sigma_{\vartheta,-\varphi}(\vartheta_1',\ -\varphi_1';\ \vartheta_2',\ -\varphi_2') \quad (22)$$

(the comparison is made between the numbers of coincidences in the last reactions of the cascade).

Since the azimuthal angles φ do not change when transferred from the center-of-mass system to the laboratory system, we can suppose that the azimuthal angles in Eqs. (1), (4), and (22) are those of the momenta of the particles in the laboratory system. We can also insert the polar angles of these momenta (referred to the same axes $\tilde{z}_c$ and $\tilde{z}_f$) instead of ϑ and ϑ', which were defined as the polar angles of the momenta $\mathbf{p}_f$ and $\mathbf{p}_i$ in the respective center-of-mass systems (i.e., instead of the numbers ϑ and ϑ' we can insert the corresponding numbers ϑ_l and ϑ_l'). Thus ϑ, φ and ϑ', φ' in Eqs. (1), (4), and (22), for example, can be taken to be the spherical angles (measured in the axes $\tilde{C}$ and $\tilde{F}$, respectively) of the particle tracks directly observed in a chamber or emulsion.

The proof of the symmetries (1), (4), and (22) for cascades involving γ-ray quanta or for cascades including reactions of decay of one particle into two particles is obtained in just the same way (all formulas and relations needed for the proof in cases involving neutrinos and photons are contained in reference 6).

4. THE TEST OF PARITY CONSERVATION AND THE AZIMUTHAL SYMMETRIES (TRIPLE SCATTERING OF PROTONS BY SPINLESS TARGETS)

Let us now pose the inverse problem: how one can experimentally test "parity conservation" in a given reaction. Strictly speaking, one must check all of the relations (14).* The validity of only part of them could be either accidental or due to some other symmetry property of the interaction (examples below).

The summetry $\sigma(\vartheta, \varphi) = (\vartheta, -\varphi)$ is a consequence of only part of the set of relations (15):

$$(0000 \mid W(\vartheta, \varphi) \mid q_c \tau_c 00) = (-1)^{q_c+\tau_c} (0000 \mid W(\vartheta, -\varphi) \mid q_c - \tau_c 00). \quad (23)$$

Conversely, if this symmetry has been confirmed experimentally and it is suggested that the relations (15) hold for the coefficients W_0, then for the reaction $c + e \to f + g$ no more is verified than this part of the whole set of relations (15) (more exactly, $2i_c$ relations). The number of the relations (21) is considerably larger, and for this reason alone we can expect that a check of (1) is a more thoroughgoing test of parity conservation. It can be objected, however, that since not all of the relations (15) are independent (see note*) it is not excluded that Eq. (23) may contain just the same number of independent relations as Eq. (21). Our assertion that Eq. (1) gives a more complete check of parity conservation that just symmetry with respect to the plane of the first reaction will now be proved by a concrete analysis of a cascade of three very simple reactions of the type of the scattering of a particle of spin $\frac{1}{2}$ by a spinless particle (for example, triple scattering of protons by helium targets).

The scattering matrix $(m' \mid R \mid m)$ (m' and m refer to different axes; cf. Sec. 1) has in this case just four elements:

$$\begin{pmatrix} (1/2 \mid R(\vartheta, 0) \mid 1/2) & (1/2 \mid R(\vartheta, 0) \mid -1/2) \\ (-1/2 \mid R(\vartheta, 0) \mid 1/2) & (-1/2 \mid R(\vartheta, 0) \mid -1/2) \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} Ae^{i\alpha} & Be^{i\beta} \\ Ce^{i\gamma} & De^{i\delta} \end{pmatrix}. \quad (24)$$

$A, B, C, D \geq 0$ and $0 \leq \alpha, \beta, \gamma, \delta \leq 2\pi$.

The relations (14) (with φ set equal to zero) reduce to just two complex equations, $a = d$, $b = -c$, if the product of the intrinsic parities of the particles does not change (as in an elastic reaction), or $a = -d$, $b = c$, if it does change sign.

According to Eqs. (5) — (8) of Sec. 1 we find the expressions for the coefficients $(q'\tau' \mid W(\vartheta, 0) \mid q\tau)$ that we shall need in terms of the elements (24):

$$(1 -1 \mid W \mid 00) \equiv (-1 \mid W \mid) = [ACe^{i(-\alpha+\gamma)} + BDe^{i(-\beta+\delta)}]/\sqrt{2}, \quad (25)$$

$$(00 \mid W \mid 1 -1) \equiv (\mid W \mid -1) = [ABe^{i(-\alpha+\beta)} + CDe^{i(-\gamma+\delta)}]/\sqrt{2}, \quad (26)$$

$$(1 -1 \mid W \mid 1 -1) \equiv (-1 \mid W \mid -1) = ADe^{i(-\alpha+\delta)}, \quad (27)$$

$$(1 -1 \mid W \mid 1 +1) \equiv (-1 \mid W \mid +1) = -BCe^{i(-\beta+\gamma)}, \quad (28)$$

$$(1 -1 \mid W \mid 10) = [ACe^{i(-\alpha+\gamma)} - BDe^{i(-\beta+\delta)}]/\sqrt{2}. \quad (29)$$

Using Eq. (16), we can write the angular distribution of the second scattering in the form

$$\sigma(\vartheta, \varphi) = \sum_{q=0}^{1} (\mid W(\vartheta, 0) \mid q0) \rho_0(q0) + 2\mathrm{Re}(\mid W(\vartheta, 0) \mid 1-1) \rho_0(1-1) \cos\varphi + 2\mathrm{Im}(\mid W(\vartheta, 0) \mid 1-1) \rho_0(1-1) \sin\varphi. \quad (30)$$

To simplify the further analysis we assume that Eq. (15) holds for the coefficients W_0 of the first reaction and W' of the third, i.e., in this sense the first reaction is the polarizer and the third is the analyzer.* Then if for even a single value of $\varphi = \varphi_0 \neq 0$ one finds that $\sigma(\vartheta, \varphi_0) = \sigma(\vartheta, -\varphi_0)$, this means that $\mathrm{Im}(\mid W \mid 1-1) \rho_0(1-1) = 0$, and thus $\mathrm{Re}(\mid W \mid 1-1) = 0$, since $\rho_0(1-1) = (1-1 \mid W_0 \mid)$ is purely imaginary by hypothesis. As can be seen from Eq. (26), the equation $\mathrm{Re}(\mid W \mid 1-1) = 0$ means the existence of a single restricting relation between the elements of the matrix R,† $\mathrm{Re}(a^*b + c^*d) = 0$, from which one cannot get the four relations $a = d$, $b = -c$. In fact, we can point out the following simple possible symmetry properties of the interaction that have the same character as the

*All of these relations are independent. If all of the spins i_c, i_d, i_a, i_b, or two of them, are half-integral, there are in all $\frac{1}{2}(2i_a + 1)(2i_b + 1)(2i_c + 1)(2i_d + 1)$ complex relations (14), or twice that many real ones. If all the spins are integral, the number of real relations is smaller by one if $\pi_c^* \pi_d^* \pi_a \pi_b = (-1)^{-i_c-i_d+i_a+i_b}$, and is larger by one if this relation does not hold. The number of the relations (15) is larger than the number of the relations (14), and therefore not all of the set (15) are independent.

*More complicated experiments (with rotation of the spin between the successive scatterings) would clearly make it possible to test parity conservation without this simplification (and provide us with polarizers and analyzers).

†Or between the coefficients K, L, M, N in the expression $R = K + L(\sigma \cdot \dot{p}' \times p) + M(\sigma \cdot p') + N(\sigma \cdot \dot{p})$ (cf. references 1, 2); that is, the symmetry $\sigma(\vartheta, \varphi) = \sigma(\vartheta, -\varphi)$ can exist even when R contains both scalar and pseudoscalar terms.

law of parity conservation and "imitate" it in the sense of making the quantity $\mathrm{Re}\,(a^*b + c^*d)$ vanish.

1) Apart from sign, the probability amplitude for the transition from the state with spin component $+\frac{1}{2}$ along the direction of the initial momentum to the state with spin component $+\frac{1}{2}$ along the final momentum is equal to the amplitude for the transition from $\frac{1}{2}$ to $-\frac{1}{2}$ (or, $a = \pm c$). The transition amplitudes $\frac{1}{2} \leftarrow -\frac{1}{2}$ and $-\frac{1}{2} \leftarrow -\frac{1}{2}$ are also equal (or $b = \mp d$).

In these terms the law of parity conservation is expressed as the equality (apart from sign) of the transition amplitudes $+\frac{1}{2} \leftarrow +\frac{1}{2}$ and $-\frac{1}{2} \leftarrow -\frac{1}{2}$ ($a = \pm d$), and $+\frac{1}{2} \leftarrow -\frac{1}{2}$ and $-\frac{1}{2} \leftarrow +\frac{1}{2}$ ($b = \mp c$).

2) $a = \pm id$, $b = \mp ic$ — the amplitudes for the corresponding transitions are equal, but unlike the case of parity conservation they differ in phase by $\pi/2$.

Doubts of parity conservation mean, in particular, that it is to be regarded as one of various possible symmetry properties (and on an equal footing with them). Only further experiments can show which of these properties exists in reality. One such experiment can be a test of the symmetry (1) in triple scattering. Let us write $\sigma_{\vartheta\varphi}(\vartheta', \varphi')$ in a form analogous to Eq. (30) [cf. Eq. (20)]:

$$\begin{aligned}\sigma_{\vartheta,\varphi}(\vartheta', \varphi') = {} & f(\vartheta', \vartheta, \vartheta_0) + 2\rho w_1' \{-\mathrm{Im}\,(-1 \mid W \mid)\cos\varphi' \\ & + \mathrm{Re}\,(-1 \mid W \mid)\sin\varphi'\} + 2w'\rho_1\{-\mathrm{Im}\,(\mid W \mid -1)\cos\varphi \\ & + \mathrm{Re}\,(\mid W \mid -1)\sin\varphi\} - 2w_1'\rho_1\{\mathrm{Re}\,(-1 \mid W \mid -1) \\ & \times\cos(\varphi' + \varphi) + \mathrm{Im}\,(-1 \mid W \mid -1)\sin(\varphi' + \varphi) \\ & + \mathrm{Re}\,(-1 \mid W \mid +1)\cos(\varphi' - \varphi) \\ & + \mathrm{Im}\,(-1 \mid W \mid +1)\sin(\varphi' - \varphi)\}, \qquad (31)\end{aligned}$$

where

$$\rho \equiv \rho_0(\vartheta_0, 00), \quad i\rho_1 \equiv \rho_0(\vartheta_0;\ 1-1),$$

$$w' \equiv (00 \mid W'(\vartheta', 0) \mid 00), \quad iw_1' \equiv (00 \mid W'(\vartheta', 0) \mid 1-1).$$

On verifying that Eq. (1) holds at four points (φ', φ), for example $(0, \pi/2)$, $(\pi/2, 0)$, $(\pi/2, \pi/2)$, $(\pi/2, -\pi/2)$, we get the results

$$\mathrm{Re}\,(-1 \mid W \mid) = \mathrm{Re}\,(\mid W \mid -1) = 0,$$

$$\mathrm{Im}\,(-1 \mid W \mid -1) = \mathrm{Im}\,(-1 \mid W \mid +1) = 0$$

or [cf. Eqs. (25) — (28)]

$$\begin{aligned}AD\sin(\alpha - \delta) = 0, \quad BC\sin(\beta - \gamma) = 0, \\ AC\cos(\alpha - \gamma) + BD\cos(\beta - \delta) = 0, \\ AB\cos(\alpha - \beta) + CD\cos(\gamma - \delta) = 0. \qquad (32)\end{aligned}$$

In addition to the "parity conservation" solutions $a = d$, $b = -c$, and $a = -d$, $b = c$, Eq. (32) also has several other solutions (for example, solutions with one or two of the parameters A, B, C, D equal to zero).

Strictly, an additional solution is needed to eliminate these other solutions. They all make the real part of the coefficient (29) vanish (which does not follow at all from parity conservation). If after the first reaction the polarization vector is subjected to a rotation (by a magnetic field, for example) around $[\mathbf{p}_c \times \mathbf{p}_l]\mathbf{p}_c^l$ so that its component in the direction of the first scattering, $\rho_0(\vartheta_0;\ 1\ 0)$, becomes different from zero, then there is added to Eq. (31) a term

$$\begin{aligned}2\rho_0(\vartheta_0;\ 10)\, w_1'\{&\mathrm{Im}\,(1-1 \mid W \mid 10)\cos\varphi' \\ &+ \mathrm{Re}\,(1-1 \mid W \mid 10)\sin\varphi'\}.\end{aligned}$$

If this destroys the symmetry (1), then $\mathrm{Re}\,(1-1\,|\,W\,|\,1\ 0) \neq 0$ and there remain the two "parity conservation" solutions. Since the intrinsic parities of the particles are not involved in Eq. (15), one cannot find out from the azimuthal symmetry whether or not the product of these particle symmetries changes.

We note that if parity conservation is established this at the same time will have the meaning that for the elastic scattering of a particle of spin $\frac{1}{2}$ by a spinless particle there is invariance under time reversal.[2]

In conclusion I express my gratitude to B. N. Valuev for a discussion of this work.

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A REFINEMENT OF THE THOMAS-FERMI MODEL AT SMALL DISTANCES

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A correction to the Thomas-Fermi model at small distances from the nucleus is suggested; this leads to a significant improvement in the agreement between the calculated and experimental values of the total energy of atoms.

IT is well known that the Thomas-Fermi model corresponds to the quasi-classical approximation to the Hartree equation. The Hartree equation has the following symbolical form in operator form:[1]

$$\nabla^2 B = -e\rho_{\text{nucl}} + \frac{8\pi e}{(2\pi\hbar)^3}\int \rho(\hat{H})\, d^3p, \qquad (1)$$

where ρ_{nucl} is the nuclear charge density ($\sim Z\delta(\mathbf{r})$), $\rho(\hat{H})$ the occupation-number operator for electrons of the form

$$\rho(\hat{H}) = (1 + \exp\{(\hat{H} - \mu)\beta\})^{-1}. \qquad (2)$$

Here $\beta = 1/kT$, T: temperature, $\hat{H}$ the Hamiltonian operator in the self-consistent approximation:

$$\hat{H} = (\mathbf{p} - i\hbar\nabla)^2/2m - eB(\mathbf{r}). \qquad (3)$$

In the quasi-classical approximation one can neglect $i\hbar\nabla$ and Eq. (1) is the same as the Thomas-Fermi equation. We get the quantum correction to the Thomas-Fermi model if we expand (2) in a power power series in $i\hbar\nabla$ acting upon $B(\mathbf{r})$. It is then essential that in the quasi-classical approximation the lower limit of the energy spectrum of $\hat{H}$ is equal to $-\infty$ which leads to an incorrect value for the density matrix (and all other quantities) at small distances from the nucleus. Using perturbation theory to take the quantum corrections approximately into account does not alter the position in this respect because the lower limit of the energy spectrum is essentially determined by all quantum corrections. However, the lower bound of the energy spectrum for atomic levels does not lie below the lowest level of a hydrogen-like atom with charge Z. This is a simple consequence of the screening influence of the atomic electrons and is confirmed experimentally. It is thus natural to attempt to generalize the occupation number operator, while not changing it in the strict sense, in such a way that also in the quasi-classical approximation the lowest level of $\hat{H}$ is bounded, and thus without violating the basic properties of the system (for instance, the virial theorem and the thermodynamic relations). This can be achieved if we propose instead of (2) for $\rho(\hat{H})$ the following expression:

$$\rho(\hat{H}) = \theta_+(\hat{H} - \mu - E_{min})(1 + \exp\{(\hat{H} - \mu)\beta\})^{-1}, \qquad (4)$$

where $\theta_+(x) = 1$ when $x > 0$, and $\theta_+(x) = 0$ when $x < 0$;

$$E_{min} \approx -e^4 m Z^2 b/2\hbar^2, \qquad 1 \leqslant b \leqslant 2.$$

If all quantum corrections are taken into account ($\hat{H} - \mu$) > E_{min} and expression (4) is equivalent to Eq. (2). At the same time, when the quantum corrections are taken into account approximately or in the quasi-classical approximation expression (4) leads to a more correct distribution for B and determines the thermodynamic characteristics of the atomic system more accurately.

Using (1) to (4) in the quasi-classical approximation we get a generalized Thomas-Fermi model. It is then expedient to go over to new variables and to introduce the following notation:

$$x = r/r_0; \qquad \varphi(x)/x = (\mu + eB)\beta, \qquad (5)$$

where r_0 is the radius of the atom and where we have introduced the notation

$$I_n(\eta, \alpha(\eta)) = \int_{\alpha(\eta)}^{\infty} y^n [1 + \exp(y - \eta)]^{-1} dy, \qquad (6)$$

$$\alpha(\eta) = (\eta - |E_{min}|\beta)\theta_+(\eta - |E_{min}|\beta). \qquad (7)$$

Using (5) to (7) we get from (1) to (4) the following equation for $\varphi(x)$ in the quasi-classical approximation:*

*In the practically most interesting case $|E_{\text{min}}|\beta \gg 1$ we have

$$I_n(\eta, \alpha(\eta)) = \begin{cases} [\eta^{n+1} - (\eta - |E_{min}|\beta)^{n+1}]/(n+1) & \text{if } \eta \geqslant |E_{min}|\beta \\ I_n(\eta, 0) & \text{if } \eta \leqslant |E_{min}|\beta. \end{cases}$$

$$\varphi''(x) = axI_{1/2}\left(\frac{\varphi(x)}{x},\ \alpha\left(\frac{\varphi(x)}{x}\right)\right) \tag{8}$$

with the boundary conditions

$$\varphi'(1) = \varphi(1), \qquad \varphi(0) = Ze^2\beta / r_0, \tag{9}$$

where

$$a = (r_0/c)^2, \qquad 1/c = 4\pi e(2m)^{3/4}/(2\pi\hbar)^{3/2}\beta^{1/4}. \tag{10}$$

The thermodynamic characteristics of the system (pressure p, average energy E) are expressed in terms of $\varphi(x)$ as follows:*

$$pV = \frac{2}{9} ZkT \frac{a}{\varphi(0)} I_{3/2}(\varphi(1),\ 0),$$

$$E = E_{kin} + E_{pot},$$

$$E_{pot} = -\frac{ZkTa}{2\varphi(0)} \int_0^1 x\,[\varphi(x) - \varphi(1)\,x + \varphi(0)]$$

$$\times I_{1/2}\left(\frac{\varphi(x)}{x},\ \alpha\left(\frac{\varphi(x)}{x}\right)\right) dx,$$

$$E_{kin} = \frac{ZkTa}{\varphi(0)} \int_0^1 x^2 I_{3/2}\left(\frac{\varphi(x)}{x},\ \alpha\left(\frac{\varphi(x)}{x}\right)\right) dx. \tag{11}$$

The virial theorem has the usual form

$$2E_{kin} + E_{pot} = 3pV,$$

V: atomic volume. In particular, we get for T = 0 from (8) (see reference 2 for the notation):

$$x^{1/2}d^2\chi/dx^2 = \chi^{3/2} - (\chi - \lambda x)^{3/2} \text{ when } x \leqslant \chi/\lambda,$$
$$x^{1/2}d^2\chi/dx^2 = \chi^{3/2} \text{ when } x \geqslant \chi/\lambda, \tag{12}$$

where

$$\lambda = 0.885\,|\,E_{min}\,|\,Z^{-4/3}. \tag{13}$$

Here $\chi(0) = 1$. The further discussion is given for the case of an isolated atom and the boundary condition is therefore of the form: $\chi'(\infty) = 0$.

As we indicated already, the quantity E_{min} must be not more than the lowest energy of a hydrogen-like atom. According to (13) we have thus $\lambda \geq 0.45\,Z^{2/3}$. If we take for λ its limiting value $\lambda = 0.45\,Z^{2/3}$; we get, solving Eq. (12), for the total energy (E) of an atom the value

$$E_{av} \approx 15.9Z^{7/3}\ \text{ev} \quad \text{for } 50 \leqslant Z \leqslant 90,$$

which agrees appreciably better with experimental values than in the case of the usual Thomas-Fermi model (in the usual Thomas-Fermi model[2] E = $20.94\,Z^{7/3}$ ev). We obtain an even better agreement with the experimental values for E, if we use for $\lambda = 0.9\,Z^{2/3}$ (which corresponds to $|E_{min}|$ equal to twice the energy of the lowest level of the hydrogen-like atom). This means that for such a choice of λ all quantum and exchange corrections to the generalized Thomas-Fermi model are minimum.

In the table we give the calculated values of E in the case where $\lambda = 0.9\,Z^{2/3}$.

Z	$E_{calc}\,Z^{-7/3}$ ev	$(E_{exp} - E_{calc})/E_{exp}$
26	15.9	0.06
36	16.2	0.053
54	16.7	0.047
80	16.9	0.05
92	17.1	0.056

We must note that a method of cutting off the energy can be applied successfully also to other problems in statistical or nuclear physics. The advantage of this method compared with cut-offs in x- or p-space lies in the fact that it corresponds to a formfactor which commutes with the total Hamiltonian and which thus does not violate the general properties of the system.

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*In deriving (11) we have used the relation

$$\frac{d}{dx} I_n(\eta(x),\ \alpha(\eta(x))) = n\,\frac{d\eta(x)}{dx}\, I_{n-1}(\eta,\ \alpha(\eta)).$$

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STRUCTURE OF A MAGNETOHYDRODYNAMIC SHOCK WAVE IN A PARTIALLY IONIZED GAS

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A magnetohydrodynamic shock wave in a partially ionized gas consists of a thin plasma discontinuity and a transition zone. The equations for the transition zone are solved approximately for certain special cases. The charge-exchange effect does not significantly influence the general character of the motion but decreases its scale. As long as the wave can be considered stationary within the transition zone the magnitude of energy dissipation is independent of the degree of ionization.

THE structure of magnetohydrodynamic shock waves in a plasma of either infinite or finite isotropic conductivity has been studied a number of times.[1-3] Viscosity and Joule heat losses determine the width of the shock front, which can be of the order of a few mean free paths or considerably smaller, depending on the shock strength and conductivity. In cases of practical interest the gas is usually not fully ionized. In interstellar gas 0.1% ionization is reached in neutral hydrogen regions and 90% in ionized hydrogen regions (helium atoms are usually neutral). Different degrees of ionization can exist in stellar atomospheres. The gas is also usually not fully ionized in laboratory experiments. It is therefore of interest to examine the structure of a wave in the presence of neutral atoms. Such a study may, in particular, help to determine when the magnetic field can serve as a "damper" which reduces shock wave dissipation.

Let a plane wave be excited by the motion of a "piston" in the direction perpendicular to a field H_0. In the absence of all interaction between ions and neutral atoms two waves would be excited by the piston — a magnetohydrodynamic wave in the plasma and an ordinary neutral gas wave. When the different kinds of particles ahead of the front have the same temperature the plasma is compressed less than the neutral gas because of magnetic pressure and the presence of electrons (the molecular weight being reduced to one-half). Therefore with identical velocity of the gases in the laboratory system the wave front will move faster in the plasma than in the neutral gas and will thus propagate in an undisturbed gas.

At temperatures below 100,000° and with not too low ionization ($\geq 10\%$) the mean free path of ions is hundreds or thousands of times smaller than the mean free path of neutral atoms. Therefore the presence of neutral atoms has no effect within the plasma wave front even when we neglect the reduction of the front width due to Joule losses, and the pressure jump is determined by the ordinary shock adiabate. The properties of the neutral gas do not become discontinuous at the shock front but will change gradually behind the front through interaction with the ions while their velocities are unequal. The present paper is a calculation of the structure of the region in which the parameters vary relatively smoothly. Joule losses and plasma viscosity are important in the much thinner plasma wave front and can be neglected in the region of present interest, which is typically hundreds of times larger. The cross section for the transfer of momentum from electrons to ions is large because of electrostatic interaction; therefore ions entrain electrons. Moreover, the inequality of their mean velocities in a plane wave would result in charge separation, which is impossible if we disregard high-frequency oscillations. Therefore the plasma can be regarded as a single whole. We shall assume for simplicity that the ion wave does not induce additional ionization. This assumption is permissible in an interstellar gas, where ionization is determined by the stellar radiation field and not by the kinetic temperature.

Under the foregoing assumptions, the equations of steady-state motion in a coordinate system moving with the shock front are given by

$$\rho_i v_i = M_i, \quad \rho_n v_n = M_n, \quad H\rho_i^{-1} = \text{const}, \tag{1}$$

$$\rho_i v_i \frac{dv_i}{dx} + \frac{dp_i}{dx} + \frac{1}{8\pi}\frac{dH^2}{dx} - (v_n - v_i)\, n_i n_n |\mu v_{\text{rel}}| \sigma = 0, \tag{2}$$

$$\rho_n v_n \frac{dv_n}{dx} + \frac{dp_n}{dx} + (v_n - v_i)\, n_i n_n \mu v_{rel} \sigma = 0, \qquad (3)$$

where M_i and M_n are constants equal to the mass flows of the respective gases, v_i and v_n are the mean velocity of the plasma and atoms, and v_{rel} is the mean relative velocity of ions and atoms which determines the collision frequency. We shall hereinafter consider not too weak waves, in which the ion velocity jump is greater than the thermal velocity; then $v_{rel} \approx v_n - v_i$. This assumption permits us to consider only the part of the wave in which the velocity difference has still not been reduced to the thermal velocity. $\mu = m_n m_i/(m_n + m_i) = m/2$ is the reduced mass (with $m_n = m_i$ for simplicity) and σ is the cross section for momentum transfer in collisions of ions with neutral atoms. p_n and p_i are determined primarily by compression and heating resulting from ion-atom collisions.

In each collision an ion and an atom acquire on the average the energy $\frac{1}{2}m(v_n - v_i)^2/4$. Therefore the heat increment (per cm^3/sec) of the ion and neutral gases is

$$q_n = q_i = {}^1/_8 n_i n_n \sigma m (v_n - v_i)^3. \qquad (4)$$

The pressure is given by

$$dp_i = \gamma \frac{p_i}{\rho_i} d\rho_i + (\gamma - 1)\, q_i dt,$$

$$dp_n = \gamma \frac{p_n}{\rho_n} d\rho_n + (\gamma - 1)\, q_n dt, \qquad (5)$$

where $dt = v_i^{-1} dx$ and $v_n^{-1} dx$, respectively.

Since neither the velocity nor the pressure remains constant behind the front our parameter of wave strength will be the velocity U of the front with respect to the quiet gas. We now introduce the dimensionless variables

$$w_i = v_i/U, \quad w_n = v_n/U, \quad \Pi_i = p_i/M_i U, \quad \Pi_n = p_n/M_n U,$$

$$Q = H_0^2/8\pi M_i U, \quad I = n_{i0}/n_{n0}, \quad \xi = n_{n0}\sigma x,$$

where n_{n0} and n_{i0} are the atom and ion concentration ahead of the front. Equations (2), (3), and (5) become

$$\frac{dw_i}{d\xi} + Q\frac{d}{d\xi}\left(\frac{1}{w_i^2}\right) + \frac{d\Pi_i}{d\xi} - \frac{1}{2}\frac{(w_n - w_i)^2}{w_i w_n} = 0, \qquad (6)$$

$$\frac{dw_n}{d\xi} + \frac{d\Pi_n}{d\xi} + I\frac{(w_n - w_i)^2}{2w_i w_n} = 0, \qquad (7)$$

$$\frac{d\Pi_i}{d\xi} + \gamma\Pi_i \frac{d\ln w_i}{d\xi} - \frac{\gamma - 1}{8}\frac{(w_n - w_i)^3}{w_i^2 w_n} = 0, \qquad (8)$$

$$\frac{d\Pi_n}{d\xi} + \gamma\Pi_n \frac{d\ln w_n}{d\xi} - \frac{\gamma - 1}{8} I \frac{(w_n - w_i)^3}{w_i w_n^2} = 0. \qquad (9)$$

At $\xi = 0$ (subscript 1) the functions are determined by the parameters of the plasma discontinuity:

$$w_{i1} = \rho_{i0}/\rho_{i1} = 1/\alpha, \quad \Pi_{i1} = p_{i1}/M_i U,$$

$$w_{n1} = 1, \quad \Pi_{n1} = \Pi_{n0} = p_{n0}/M_n U.$$

Compression and heating in a magnetohydrodynamic wave have previously been calculated,[4,5] but must now be expressed in our dimensionless quantities. From the shock adiabate for a monatomic gas we obtain

$$Q\alpha^2 + 5\alpha(Q + \Pi_{i0} + 0.2) - 4 = 0, \qquad (10)$$

$$\Pi_{i1} - \Pi_{i0} = (\alpha - 1)[1/\alpha - Q(\alpha + 1)]. \qquad (11)$$

Equations (6) — (9) can be integrated numerically but the basic parameters of the solution can be obtained in the rough approximation $\Pi_i = \Pi_n = 0$. Subsequent estimates indicate that this approximation is adequate in many cases since, although the wave velocity may considerably exceed the velocity of sound, heating of the gas is insignificant in the presence of a sufficiently strong field.[6] This is evident, specifically, from the adiabat for the gas as a whole.

Multiplying (6) by I, adding to (7) and integrating, we obtain the momentum integral for the entire gas:

$$w_i + Q/w_i^2 + \Pi_i + w_n/I + \Pi_n/I$$

$$= w_{i1} + Q/w_{i1}^2 + \Pi_{i1} + w_{n1}/I + \Pi_{n1}/I$$

$$= 1/\alpha + Q\alpha^2 + 1/I + \Pi_{i1} + \Pi_{n1}/I. \qquad (12)$$

Since α and Q are related by (10) and we have assumed $\Pi_{n1} = \Pi_{i1} = 0$, (12) gives us w_i in terms of w_n and I as well as the solution for a rarefaction wave. ξ for a pair of values of w_n and w_i is obtained from (7) by means of a quadrature:

$$\xi = -\frac{2}{I}\int_1^{w_n} \frac{w_n w_i}{(w_n - w_i)^2} dw_n. \qquad (13)$$

The result of the calculation for Q = 0.25 and I = 1 is shown in Fig. 1. This example corresponds to the average conditions expected in the rarefied gas of our galaxy[7] — $\rho_{n0} \approx \rho_{i0} \approx 10^{-26}$ g/cm^3, v = 100 km/sec, $H \approx 5 \times 10^{-6}$ oersteds, U ≈ 200 km/sec. U is taken to be twice as large as v,

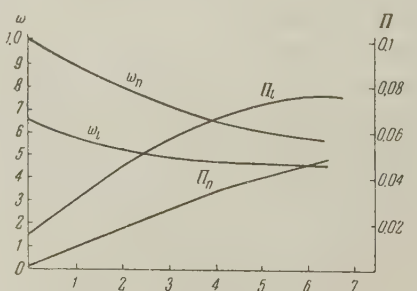

FIG. 1. Q = 0.25, I = 1.

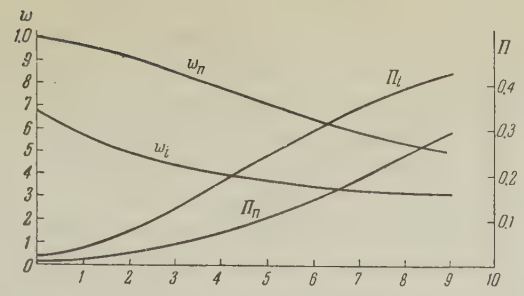

FIG. 2. Q = 0.25, I = 0.3.

since, as is evident from Fig. 1, the final gas compression is close to 2 ($w \approx 0.5$).

In order to calculate the pressures Π_i and Π_n we assume in first approximation that pressure has little effect on the motion. Substituting the values of w_i and w_n calculated above into (8) and (9) and performing a numerical integration, we obtain the curves of Π_i and Π_n in Fig. 1. We have $\Pi_{n,i} \ll w_{n,i}$; therefore, as is evident from (12), the first approximation is adequate and the motion actually depends only slightly on the gas pressure.

In order to elucidate the dependence of the solution on the parameters similar calculations were carried out for the following cases: Q = 0.25, I = 0.3; Q = 0.25, I = 3; Q = 0.15, I = 1 (Figs. 2 — 4). The general character of the solution is the same in all cases, but in a weak field and with strong ionization w_n varies considerably in the distance $\xi = 1$, so that the ordinary viscosity of the neutral gas may now be important. With weak ionization Π_i has a considerable effect on the motion since with low ionization the field moderates the compression to a lesser extent ($w_i \approx 0.3$) and supersonic motion results in considerable heating of the gas as a whole.

We now calculate the irreversible dissipation of energy in the wave. Let the wave compress and heat the gas, after which it expands adiabatically to its initial density in a long time interval. The internal energy of a mass unit in the wave is given for both the plasma and gas by

$$\frac{1}{\gamma-1}\frac{p}{\rho} = \frac{1}{\gamma-1}\Pi w U^2$$

with the proper subscripts. During the subsequent adiabatic expansion this energy varies proportionally to $w^{\gamma-1}$. Thus the irreversible energy dissipation amounts to $U^2(\gamma-1)^{-1}(\Pi_i w_i^\gamma + \Pi_n w_n^\gamma) - \epsilon_0$, where ϵ_0 is the internal energy ahead of the front. This quantity can be easily computed by means of Figs. 1 — 4. In all of the cases considered except that of a strongly ionized gas (I = 3) most of the dissipation occurs behind the plasma discontinuity rather than at the discontinuity. For I = 3 the dissipation at the discontinuity is approximately equal to that behind it.

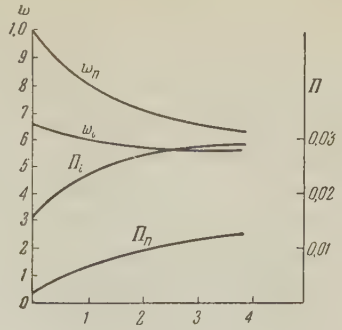

FIG. 3. Q = 0.25, I = 3.

The irreversible energy dissipation, the final total pressure and final velocity can also be determined by the shock adiabate method[5] if we assume that there is sufficient friction between the gases and that the magnetic pressure thus also acts on the neutral gas. The figures show that for this purpose the separation of the planes at which jumps of the various quantities are measured must exceed 5 or 10 mean free paths in the neutral gas. The calculation can be preformed by means of (10) and (11) with Q and Π_i replaced by Q′ and Π′ as follows:

$$Q' = \frac{H_0^2}{8\pi(M_i+M_n)U} = Q\frac{I}{1+I},$$

$$\Pi' = \frac{p_n+p_i}{(M_n+M_i)U} = \frac{\Pi_n+I\Pi_i}{1+I}.$$

The results agree with Figs. 1 — 4 to within 10 — 20%. Where the influence of Π_i on motion can no longer be neglected Fig. 1, as would be expected, gives systematically larger values than the shock adiabate. Since Q is smaller for the entire gas than for the plasma alone the final compression must be greater than at the plasma discontinuity. This accounts for the fact that the curve of w_i continues to descend and does not approach the curve of w_n.

We have thus far only considered the usual elastic interaction between ions and atoms, which is sufficient if these do not belong to the same element and their relative velocities are not too large

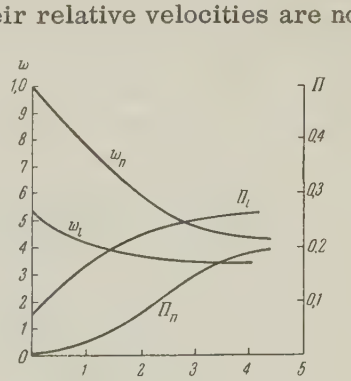

FIG. 4. Q = 0.15, I = 1.

(below 1000 km/sec, for example). When these conditions are not fulfilled charge exchange becomes much more significant; an electron is transfered from an atom to an ion without essential change in the motion of either particle. The charge-exchange cross sections for identical particles is larger than the gaskinetic cross section by a factor of a few tens. The width of the transition zone will be correspondingly reduced but will still exceed that of the plasma wave front (when Joule losses are taken into account), and the division of the wave into a plasma discontinuity and a transition zone remains valid. Therefore the fundamental equations will not be changed greatly if the charge-exchange cross section σ_i is introduced into the definition $\xi = n_{n0}\sigma x$. The principal difference will lie in the fact that after charge exchange the velocities of the particles will remain practically unchanged; this is equivalent to 180° scattering. Therefore the momentum change of the gases in one charge exchange will be given by $m(v_n - v_i)$ rather than by $\frac{1}{2}m(v_n - v_i)$ as previously; this means that the last terms in (2) and (3) are doubled.

The quantities q_n and q_i in (4) are also changed. The neutral atom resulting from charge exchange retains the thermal component of ion velocity, so that charge exchange is a mechanism for heat transfer. The resultant ion possesses the velocity $v_n - v_i$ with respect to the plasma and its associated field. The energy of relative motion is slowly transformed into thermal energy through collisions with other ions if the mean free ion path is smaller than the radius of gyration. When the mean free ion path is greater than the radius of gyration, during a single rotation this energy is transformed into the energy of spiral motion and then, as a result of collisions, into thermal energy. We can therefore write

$$q_i = n_i n_n \sigma_1 [{}^1/_2 m (v_n - v_i)^2 - {}^3/_2 k (T_i - T_n)] (v_n - v_i),$$
$$q_n = {}^3/_2 n_i n_n \sigma_1 k (T_i - T_n)(v_n - v_i). \qquad (14)$$

Since there are practically no collisions between neutral atoms during the time between charge exchanges the distribution of the neutral gas cannot be Maxwellian. However, since atoms are produced from ions, which can exchange energy at not too high temperatures, the distribution of atomic velocities will not be extremely non-Maxwellian. Deviations can result from 1) the continuous rise of the plasma "temperature" in conjunction with different atom "ages" and 2) the departure of the ion velocity distribution from the Maxwellian because of the limited interaction time. This is especially important for relatively fast particles since the elastic ion-scattering cross section for ions decreases as v^{-4}. Therefore the thermal velocities of individual particles can hardly exceed $v_n - v_i$. On the other hand, $v_n - v_i$ decreases continually, thus complicating the picture to an even greater degree.

Returning to the basic equations, we find that the first equation in (5) will retain its form with a different meaning for q_i. The second equation is changed since there is no adiabatic heating in the absence of collisions between atoms. The atom concentration increases because there is an increase in the concentration of the ions out of which the former are produced, but the atoms themselves are subject to no forces. Therefore in the second equation of (5) only the second term remains, with the suitable meaning of q_n. Equations (6) and (7) remain unchanged with $\xi = 2n_{n0}\sigma_i x$, which means that in a single charge exchange twice as much momentum is transferred as previously.

It would not be meaningful to solve the complete system of equations under the given simplifying assumptions. We shall consider only the heat supplied to both gases:

$$q = q_i + q_n = {}^1/_2 n_i n_n \sigma_1 m (v_n - v_i)^3,$$

which is twice as large as previously. Both the rate of heating and the rate of momentum transfer have been doubled. Therefore the entire process occurs twice as rapidly (taking the mean free time as the unit) but ultimately results in the same change of total internal gas energy as previously. This also follows, of course, from the fact that the shock adiabat must give the same solution for the sum of the gases independently of their interaction mechanism.

Summarizing, it can be stated that when pressure plays an insignificant part the velocity distribution will be that shown in the figures but with the new definition of ξ. The values of Π_i and Π_n will be somewhat changed but the general character of the curves and their sum will be conserved. The curves of Π_n and Π_i will be separated by a distance corresponding to the mean time for charge exchange, which serves as the mechanism for transferring energy from ions to atoms. At high gas velocities the ion path before elastic scattering by an ion can become greater than its path before charge exchange, and Joule losses will be small because of the high temperature. In this case a plasma wave front will not be formed; the plasma and atoms will move together with the field at some mean velocity. Momentum transfer will still occur as a result of charge exchange and ion acceleration

by the field. The entropy will increase because the field twists the trajectories of ions passing from the quiet gas through the front as neutral atoms, after which collisions occur, and also because of the Joule losses.

Since the energy dissipation of the wave in a partially ionized gas is given by the shock adiabate with Q for the entire gas, it does not depend on the degree of ionization when the wave can be regarded as stationary within the transition region. With extremely low ionization the transition region may become so wide that this condition is not fulfilled. However, when charge exchange occurs the transition zone is less wide and the possibility of nonstationary conditions plays a smaller part.

I am deeply grateful to Ya. B. Zel'dovich, who read the manuscript and made a number of valuable comments.

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SCATTERING OF DIRAC PARTICLES IN THE SECOND BORN APPROXIMATION

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The problem treated is that of the elastic scattering of Dirac particles by a fixed spherically symmetrical center of force. The values of the scattering amplitudes are found in the second Born approximation.

IN a paper by Sokolov and the writers[1] values of the scattering phase shifts $\delta_l^{(1)}$ and $\delta_l^{(2)}$ for the scattering of Dirac particles by an arbitrary force center were found in second approximation in the interaction potential [see Eq. (23) of reference 1].

Using the following integral representations of the spherical Bessel functions:

$$j_l^2(kr) = \frac{1}{2kr}\int_0^{\pi} \sin\left(2kr\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,P_l(\cos\varphi)\,d\varphi,$$

$$j_l(kr)\,n_l(kr) = -\frac{1}{2kr}\int_0^{\pi} \cos\left(2kr\sin\frac{\varphi}{2}\right)\cos\frac{\varphi}{2}\,P_l(\cos\varphi)\,d\varphi,$$

we can put the expressions in question in the form

$$\tan\delta_l^{(1)} = -\frac{K}{c\hbar}\int_0^{\pi}[\alpha P_l(\cos\varphi) + \beta P_{l+1}(\cos\varphi)]\,L(\varphi)\,d\varphi$$
$$+\left(\frac{K}{c\hbar}\right)^2\int_0^{\pi}\int_0^{\pi}[\alpha^2 P_l(\cos\varphi)\,P_l(\cos\psi)$$
$$+\alpha\beta P_l(\cos\varphi)\,P_{l+1}(\cos\psi) + \alpha\beta P_{l+1}(\cos\varphi)\,P_l(\cos\psi)$$
$$+\beta^2 P_{l+1}(\cos\varphi)\,P_{l+1}(\cos\psi)]\,M(\varphi,\psi)\,d\varphi\,d\psi,$$

$$\tan\delta_l^{(2)} = -\frac{K}{c\hbar}\int_0^{\pi}[\alpha P_l(\cos\varphi) + \beta P_{l-1}(\cos\varphi)]\,L(\varphi)\,d\varphi$$
$$+\left(\frac{K}{c\hbar}\right)^2\int_0^{\pi}\int_0^{\pi}[\alpha^2 P_l(\cos\varphi)\,P_l(\cos\psi)$$
$$+\alpha\beta P_l(\cos\varphi)\,P_{l-1}(\cos\psi) + \alpha\beta P_{l-1}(\cos\varphi)\,P_l(\cos\psi)$$
$$+\beta^2 P_{l-1}(\cos\varphi)\,P_{l-1}(\cos\psi)]\,M(\varphi,\psi)\,d\varphi\,d\psi, \qquad (1)$$

where

$$L(\varphi) = \frac{1}{2}\cos\frac{\varphi}{2}\int_0^{\infty}\sin\left(2kr\sin\frac{\varphi}{2}\right)V(r)\,r\,dr,$$

$$M(\varphi,\psi) = \frac{1}{2}\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\int_0^{\infty}\cos\left(2kr\sin\frac{\varphi}{2}\right)V(r)\,r\,dr$$
$$\times\int_0^{r}\sin\left(2kr'\sin\frac{\psi}{2}\right)V(r')\,r'\,dr'. \qquad (2)$$

The rest of the notation is that of reference 1. In particular, $\hbar k$ is the momentum, $E = c\hbar K$ is the energy, and $m = \hbar k_0/c$ is the mass of the particle.

Assuming the phase shifts small, we set $\tan\delta_l \approx \delta_l$, and instead of the exact expressions for the scattering amplitudes $f(\vartheta)$ and $g(\vartheta)$ we content ourselves with the linear approximations

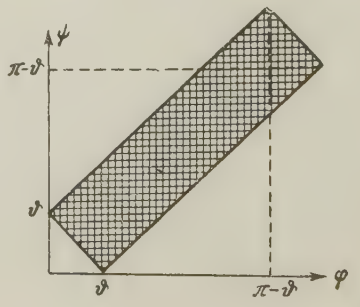
$$f(\vartheta) = k^{-1}\sum_{l=0}^{\infty}[(l+1)\,\delta_l^{(1)} + l\delta_l^{(2)}]\,P_l(\cos\vartheta),$$

$$g(\vartheta) = k^{-1}\sum_{l=1}^{\infty}[\delta_l^{(1)} - \delta_l^{(2)}]\,P_l^1(\cos\vartheta),$$

where $P_l^1(\cos\vartheta) = -\sin\vartheta\,dP_l(\cos\vartheta)/d\cos\vartheta$, and $P_l(\cos\vartheta)$ is the Legendre polynomial. Substituting here the values of $\delta_l^{(1)}$ and $\delta_l^{(2)}$ from Eq. (1) and carrying out the summation over l by the use of Eqs. (8a) — (8e) of the Appendix, we get

$$f(\vartheta) = -\frac{K}{c\hbar k}\frac{\alpha+\beta\cos\vartheta}{2\sin\left(\frac{\vartheta}{2}\right)}\int_0^{\infty}\sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr$$
$$+\frac{2K^2}{\pi c^2\hbar^2 k}\iint_{\Omega}[\alpha^2 + \alpha\beta(\cos\varphi+\cos\psi)$$
$$+\beta^2\cos\vartheta]\,R^{-1}M(\varphi,\psi)\,d\varphi\,d\psi; \qquad (3)$$

$$g(\vartheta) = -\frac{K}{c\hbar k}\frac{\beta\sin\vartheta}{2\sin\left(\frac{\vartheta}{2}\right)}\int_0^{\infty}\sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr$$
$$+\frac{2K^2}{\pi c^2\hbar^2 k}\iint_{\Omega}\left[\alpha\beta\tan\frac{\vartheta}{2}(\cos\varphi+\cos\psi)\right.$$
$$\left.+\beta^2\sin\vartheta\right]R^{-1}M(\varphi,\psi)\,d\varphi\,d\psi, \qquad (4)$$



where the region Ω is defined by the conditions (9) and shown graphically in the diagram, and

$$R = [1 - \cos^2\varphi - \cos^2\psi - \cos^2\vartheta + 2\cos\varphi\cos\psi\cos\vartheta]^{1/2}. \quad (4')$$

In the particular case of Coulomb scattering $V(r) = -Ze^2/r$ and

$$M^c(\varphi, \psi) = \frac{\pi Z^2 e^4}{8k}\cos\frac{\varphi}{2}\cot\frac{\psi}{2}\left\{2\delta\left(2k\sin\frac{\varphi}{2}\right) - \delta\left[2k\left(\sin\frac{\varphi}{2} - \sin\frac{\psi}{2}\right)\right] - \delta\left[2k\left(\sin\frac{\varphi}{2} + \sin\frac{\psi}{2}\right)\right]\right\}.$$

Using this, we can get the well known formula of McKinley and Feshbach (cf., e.g., reference 1) from Eqs. (3) and (4). If various special spherically symmetrical charge distributions inside the nucleus are specified, one can calculate by numerical integration the deviations from pure Coulomb scattering caused by the finite dimensions of the nucleus.

A particularly interesting case is that of high energies, for which one can neglect the rest mass of the particle in the Dirac equation in comparison with its total energy. This does not make any important change in the picture of the scattering, and in the final results it affects only terms of the order $(mc^2/E)^2$. It can be shown that when the rest mass is neglected the phase shifts corresponding to a prescribed total angular momentum are exactly equal,[2] i.e., $\delta_l^{(1)} = \delta_{l+1}^{(2)}$. In our case this can be seen directly from the expressions (1) and (2) if we set $\alpha = \beta = 1$. Using this fact, one can easily show that when the rest mass is neglected

$$g_{ur}(\vartheta) = \tan\left(\frac{\vartheta}{2}\right) f_{ur}(\vartheta),$$

and the differential cross-section takes the form

$$d\sigma_{ur}/d\Omega = \sec^2(\vartheta/2)\,|\,f_{ur}(\vartheta)\,|^2, \quad (5)$$

where

$$f_{ur}(\vartheta) = -\frac{K}{c\hbar k}\frac{\cos^2(\vartheta/2)}{\sin(\vartheta/2)}\int_0^\infty \sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr + \frac{2K^2}{\pi c^2\hbar^2 k}\iint_\Omega (1 + \cos\varphi + \cos\psi + \cos\vartheta)\,R^{-1}M(\varphi, \psi)\,d\varphi\,d\psi.$$

When we go to the nonrelativistic case we must set $\alpha = 1$, $\beta = 0$, and we have from Eqs. (3) and (4)

$$f_{nr}(\vartheta) = -\frac{K}{c\hbar k}\frac{1}{2\sin(\vartheta/2)}\int_0^\infty \sin\left(2kr\sin\frac{\vartheta}{2}\right)V(r)\,r\,dr + \frac{2K^2}{\pi c^2\hbar^2 k}\iint_\Omega R^{-1}M(\varphi, \psi)\,d\varphi\,d\psi,$$

$$g_{nr}(\vartheta) = 0. \quad (6)$$

It can be shown that in the case of pure Coulomb scattering in nonrelativistic approximation the second term in the expression (6) for f_{nr} makes the contribution zero. This is in agreement with the fact that the classical Rutherford formula is exact within the framework of nonrelativistic quantum mechanics.

APPENDIX

By means of the formalism of the Dirac δ function one can calculate a number of sums containing products of three Legendre polynomials. We construct a δ function from the orthonormal Legendre polynomials:

$$\sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)P_l(\cos\omega)\,P_l(\cos\vartheta) = \delta(\cos\omega - \cos\vartheta) \quad (7)$$

and set

$$\cos\omega = \cos\varphi\cos\psi + \sin\varphi\sin\psi\cos\gamma.$$

Using the fact that

$$\frac{1}{\pi}\int_0^\pi P_l(\cos\omega)\,d\gamma = P_l(\cos\varphi)\,P_l(\cos\psi),$$

we integrate Eq. (7) from 0 to π. Examination of the limits of integration and use of the fundamental property of the δ function give the following value for the sum of products of three Legendre polynomials:

$$\sum_{l=0}^{\infty}\left(l + \tfrac{1}{2}\right)P_l(\cos\varphi)\,P_l(\cos\psi)\,P_l(\cos\vartheta) = \begin{cases} 1/\pi R & \text{inside } \Omega, \\ 0 & \text{outside } \Omega, \end{cases} \quad (8a)$$

where the region Ω is defined by the conditions

$$\varphi + \psi + \vartheta \leqslant 2\pi, \quad \varphi + \psi - \vartheta \geqslant 0, \quad \varphi - \psi + \vartheta \geqslant 0, \quad -\varphi + \psi + \vartheta \geqslant 0 \quad (9)$$

and has the shape shown in the diagram, and R is given by Eq. (4').

Using the well known recurrence relations between the Legendre polynomials, we can get from Eq. (8a) the values of the following sums:

$$\sum_{l=0}^{\infty} P_l(\cos\varphi)\,[(l+1)\,P_{l+1}(\cos\psi) + lP_{l-1}(\cos\psi)]\,P_l(\cos\vartheta) = \begin{cases} 2\cos\psi/\pi R & \text{inside } \Omega, \\ 0 & \text{outside } \Omega; \end{cases} \quad (8b)$$

$$\sum_{l=0}^{\infty}[(l+1)\,P_{l+1}(\cos\varphi)\,P_{l+1}(\cos\psi) + lP_{l-1}(\cos\varphi)\,P_{l-1}(\cos\psi)]\,P_l(\cos\vartheta) = \begin{cases} 2\cos\vartheta/\pi R & \text{inside } \Omega, \\ 0 & \text{outside } \Omega. \end{cases} \quad (8c)$$

Starting from the value of the sum

$$\sum_{l=1}^{\infty} [P_{l+1}(\cos\omega) - P_{l-1}(\cos\omega)]\, P_l^1(\cos\vartheta)$$
$$= 2\sin\vartheta\,\delta(\cos\omega - \cos\vartheta),$$

we can calculate by an analogous method the following sums:

$$\sum_{l=1}^{\infty} P_l(\cos\varphi)\,[P_{l+1}(\cos\psi) - P_{l-1}(\cos\psi)\, P_l^1(\cos\vartheta)$$
$$= \begin{cases} 2(\cos\varphi - \cos\psi\cos\vartheta)/\pi R\sin\vartheta & \text{inside } \Omega, \\ 0 & \text{outside } \Omega; \end{cases} \quad (8d)$$

$$\sum_{l=1}^{\infty} [P_{l+1}(\cos\varphi)\, P_{l+1}(\cos\psi) - P_{l-1}(\cos\varphi)\, P_{l-1}(\cos\psi)]$$
$$\times P_l^1(\cos\vartheta) = \begin{cases} 2\sin\vartheta/\pi R & \text{inside } \Omega, \\ 0 & \text{outside } \Omega. \end{cases} \quad (8e)$$

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SOME APPLICATIONS IN THE THEORY OF METALS OF THE METHOD OF SUMMATION OF THE MAIN FEYNMAN DIAGRAMS

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Starting from the condition that the average phonon energy must be small compared to the average energy of the electron transitions, we have constructed a Hamiltonian with a direct electron-electron interaction which describes in the given approximation a Fröhlich system of interacting electrons and phonons.

THE Fröhlich model has recently received widespread interest; this model is described by the Hamiltonian

$$H = H_{el} + H_{ph} + H_{int};$$
$$H_{el} = \sum_{ks} \varepsilon(k)\, a^+_{k,s} a_{k,s}; \qquad H_{ph} = \sum_q \omega(q)\, b^+_q b_q;$$
$$H_{int} = \sum_{kqs} g(q) \sqrt{\frac{\omega(q)}{2V}}\, a^+_{k,s} a_{k+q,s} (b^+_q + b_{-q}). \qquad (1)$$

A number of important physical results can, however, be obtained from a simpler model without phonons, the Hamiltonian of which is of the form:[1,2]

$$H_1 = H_{el} + H'_{int};$$
$$H'_{int} = -\frac{1}{2V} \sum_{kk'qss'} I(kk'q)\, a^+_{k,s} a_{k+q,s} a^+_{k',s'} a_{k'-q,s'}, \qquad (2)$$

where I(kk'q) is a function which decreases sufficiently fast when one gets away from the Fermi surface. The conditions, however under which the models described by (1) and (2) can be considered to be approximately the same are still not completely clear, notwithstanding a number of studies which have been made.[1,2] Moreover, there does not exists a sufficiently consistent way of determining the function I(kk'q) in (2).

The method of summing the main Feyman diagrams makes it possible to construct the Hamiltonian (2) from the Hamiltonian (1) under only one assumption:

$$\bar{\omega} \ll \overline{\Delta\varepsilon} \qquad (3)$$

(average phonon energy negligibly small compared to the average energy of the electron transitions).

Performing the usual procedure of renaming the creation and annihilation operators of the electrons inside the Fermi sphere, we go after that over to the "interaction representation" where the part of the Hamiltonian of non-interacting fields will be played by the quadratic form

$$H_0 = \sum_q \omega(q)\, b^+_q b_q + \sum_{ks} \tilde{\varepsilon}(k)\, a^+_{k,s} a_{k,s},$$

where

$$\tilde{\varepsilon}(k) = \begin{cases} \varepsilon(k) & (|k| > k_F) \\ -\varepsilon(k) & (|k| < k_F). \end{cases} \qquad (4)$$

In the following we shall in accordance with reference 2 give our discussion along the lines of the covariant formulation of the quantum theory of fields. The S matrix of the system is of the form

$$S = S^0_{-\infty} = T\left[\exp\left(-i \int_{-\infty}^{0} H_{int}(\theta)\, d\theta\right)\right]. \qquad (5)$$

The energy of the interaction of the electrons with the phonons, which does not depend on the time, is, if we take all radiative corrections into account, equal to

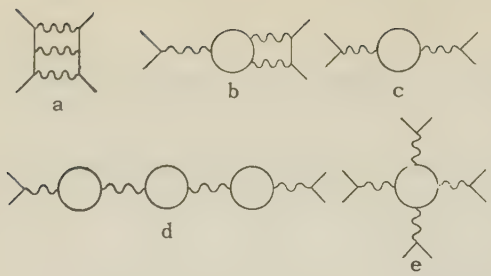
$$\hat{R} = T\left[H_{int}(0) \exp\left(-i \int_{-\infty}^{0} H_{int}(\theta)\, d\theta\right)\right]. \qquad (6)$$

The chronological pairs which occur when $\hat{R}$ is written out explicitly are written in the usual way:

$$\dot{B}_q(t)\, \dot{B}_{q'}(t') = [\theta(t-t')\, e^{i\omega(q)(t'-t)} + \theta(t'-t)\, e^{i\omega(q)(t-t')}]\, \delta_{q+q',0};$$
$$B_q(t) = b^+_q(t) + b_{-q}(t);$$
$$\dot{a}_{k,s}(t)\, \dot{a}^+_{k',s'}(t') = -\dot{a}^+_{k',s'}(t')\, \dot{a}_{k,s}(t) = \delta_{kk'}\delta_{ss'} \begin{cases} \theta(t-t')\, e^{-i\tilde{\varepsilon}(k)(t-t')} & (|k|>k_F) \\ -\theta(t'-t)\, e^{i\tilde{\varepsilon}(k)(t-t')} & (|k|<k_F) \end{cases} \qquad (7)$$

(to attain more compact formulae we have marked the operators corresponding to a chronological pairing everywhere by the same number of dots on top).

The effective Hamiltonian with a direct electron-electron interaction is a sum of terms, each of which is described by a Feynman diagram with external electron lines (diagrams of the kind illus-



trated in the figure). Part of the diagrams corresponds to triple, quadruple, ... interactions between electrons. We shall retain in the operator series $\hat{R}$ the terms with coefficients of the lowest order in ω, i.e., such terms where the number of integrations cancelling in the denominator 2ω, is as large as possible for a given order in g^2. This leads to the following basic rule of selecting the main diagrams: in each term of the expansion of (6) in g^2 those diagrams must remain for which the number of possible ways of breaking them into two parts by lines intersecting only phonon lines is as large as possible. We shall denote the contribution of the diagrams with external electron lines by $\hat{R}^{(el)}$ and we shall in the following use (3) to look for a formal approximate expression for $\hat{R}^{(el)}$. It is clear that one can at once drop diagrams of the type a, b, in the Fig.

Using the relations

$$\omega(-q) = \omega(q), \qquad g(-q) = g(q) \tag{8}$$

all integrations over the time in terms corresponding to diagrams of the type c, d, and e can be performed simply enough explicitly and the result of each integration is the appearance of the corresponding energy denominator. This makes it possible to formulate two more rules: one must discard all diagrams with more than four external electron lines (type e), and from diagrams of the type c or d only retain those for which the pair of external electron lines starts from a vertex corresponding to the first factor under the T-product sign in (6).

The correctness of all rules mentioned is illustrated below by the example of evaluating the first three terms in the expansion of $\hat{R}^{(el)}$ in g^2. We have

$$\hat{R}_0^{(el)} = -\frac{i}{2V}\int_{-\infty}^{0} dt \sum_{kk'qq'ss'} g(q)\, g(q')\, V\overline{\omega(q)\,\omega(q')}\, a^+_{k,s}(0)$$
$$\times a_{k+q,s}(0)\, \dot{B}_q(0)\, a^+_{k',s'}(t)\, a_{k'+q',s'}(t)\, \dot{B}_{q'}(t)$$
$$= -\frac{1}{2V}\sum_{kk'qss'} \frac{g^2(q)\,\omega^2(q)}{-z^2(k',q)+\omega^2(q)}$$
$$\times a^+_{k,s}(0)\, a_{k+q,s}(0)\, a^+_{k',s'}(0)\, a_{k'-q,s'}(0); \tag{9}$$

$$z(k',q) = \varepsilon(k') - \varepsilon(k'-q); \tag{10}$$

(the N-product signs are here and everywhere in the following omitted).

In evaluating the further terms we shall all the time meet with the factor

$$C(t',t) = \int_{-\infty}^{0} dt_1 \sum_{k_1k_2s_1s_2q_1q_2} \dot{B}_{q'}(t')\, \ddot{a}^+_{k_1,s_1}(t_1)$$
$$\times \dddot{a}_{k_1+q_1,s_1}(t_1)\, \dot{B}_{q_1}(t_1)\, \dddot{a}^+_{k_2,s_2}(t)\, \ddot{a}_{k_2+q_2,s_2}(t)\, B_{q_2}(t)$$
$$= 2\sum_q \sum_k^{(q)} \left\{ -\frac{e^{i\omega t'+iYt}}{i(\omega+Y)} + \frac{2\theta(t'-t)}{i(Y^2-\omega^2)}\left[Ye^{i\omega(t-t')} - \omega e^{iY(t-t')}\right] \right.$$
$$\left. + \frac{2\theta(t-t')}{i(Y^2-\omega^2)}\left[Ye^{i\omega(t'-t)} - \omega e^{iY(t'-t)}\right] \right\} B_{-q}(t)$$
$$= 2\sum_k^{(q)} C_{k,q}(t',t)\, B_{-q}(t);$$
$$\omega = \omega(q), \qquad Y = Y(k,q) = \tilde{\varepsilon}(k+q) + \tilde{\varepsilon}(k). \tag{11}$$

The symbol $\sum_k^{(q)}$ indicates a summation over k where $|k+q| > k_F$; $|k| < k_F$.

We shall now evaluate the term $\hat{R}_1^{(el)}$ which corresponds to a diagram with one electron-hole loop (type c):

$$\hat{R}_1^{(el)} = \int_{-\infty}^{0}\int_{-\infty}^{0} dt\, dt_1 \sum_{kk'qq'ss'} \frac{g^3(q)\, g(q')\, V\overline{\omega^3(q)\,\omega(q')}}{(2V)^2}\, a^+_{k,s}(0)$$
$$\times a_{k+q,s}(0)\, a^+_{k',s'}(t_1)\, a_{k'+q',s'}(t_1)\, \dot{B}_{q'}(t_1)\, \dot{C}(0t)$$
$$\approx -\frac{2}{(2V)^2}\sum_{kk'qss'} \frac{g^4(q)\,\omega^2(q)}{-z^2(k',q)+\omega^2(q)} \sum_{k_1}^{(q)} \frac{1}{\tilde{\varepsilon}(k_1)+\tilde{\varepsilon}(k_1-q)}$$
$$\times a^+_{k,s}(0)\, a_{k+q,s}(0)\, a^+_{k',s'}(0)\, a_{k'-q,s'}(0). \tag{12}$$

The term $\hat{R}_2^{(el)}$ describing the diagram with two electron-hole loops is equal to

$$\hat{R}_2^{(el)} = 4\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{-\infty}^{0} dt_1 dt_2 dt_3$$
$$\times \sum_{kk'qq'ss'} \sum_{k_1}^{(q)} \sum_{k_2}^{(-q)} \frac{g^5(q)\, g(q')\, V\overline{\omega^5(q)\,\omega(q')}}{(2V)^3}\, a^+_{k,s}(0)\, a_{k+q,s}(0)$$
$$\times a^+_{k's'}(t_3)\, a_{k'+q',s'}(t_3)\, \dot{B}_{q'}(t_2)\, \dot{B}_{q'}(t_3)\, C_{k_1,q}(0,t_1)\, C_{k_2,-q}(t_1,t_2)$$
$$\approx -\frac{8}{(2V)^3}\sum_{kk'qss'} \frac{g^6(q)\,\omega^2(q)}{-z^2(k',q)+\omega^2(q)} \left[\sum_{k_1}^{(q)} \frac{1}{\tilde{\varepsilon}(k_1)+\tilde{\varepsilon}(k_1-q)}\right]^2$$
$$\times a^+_{k,s}(0)\, a_{k+q,s}(0)\, a^+_{k',s'}(0)\, a_{k'-q,s'}(0). \tag{13}$$

Apart from the terms described by Eqs. (12) and (13), $\hat{R}_1^{(el)}$ and $\hat{R}_2^{(el)}$ contain a number of terms with higher powers of ω in the numerator. One notes easily that the subsequent terms in the series expansion of $\hat{R}^{(el)}$ must differ from the

preceding ones by an increase in the power of the expression

$$2A(q) = \frac{2g^2(q)}{V} \sum_{k_1}^{(q)} \frac{1}{\tilde{\varepsilon}(k_1) + \tilde{\varepsilon}(k_1 - q)} \tag{14}$$

under the summation sign. The final expression for $\hat{R}^{(el)}$ is thus in the approximation (3) of the form

$$\hat{R}^{(el)} = -\frac{1}{2V} \sum_{kk'q'ss'} \frac{g^2(q)\,\omega^2(q)\,\Xi(A)}{-z^2(k', q) + \omega^2(q)}$$

$$\times a^+_{k,s}(0)\, a_{k+q,s}(0)\, a^+_{k',s'}(0)\, a_{k'-q,s'}(0), \tag{15}$$

$$\Xi(A) = 1 + A + 2A^2 + 4A^3 + \ldots$$

$$= (1 - A)/(1 - 2A). \tag{16}$$

The function I(kk'q) from (2) is according to (15) and (16) determined by the formula

$$I(kk'q) = \frac{g^2(q)\,\omega^2(q)}{-[\varepsilon(k') - \varepsilon(k' - q)]^2 + \omega^2(q)} \frac{1 - A(q)}{1 - 2A(q)}. \tag{17}$$

One can similarly sum the diagrams with external phonon lines and renormalize the phonon energy (although the result needs more unwieldy calculations).

The calculations given here make it possible to establish an inequality which must be satisfied by the constant

$$2\max A = g^2 (dn/dE)_{E=E_F} = \rho.$$

Indeed, the convergence of the series (16) requires that $\rho < 1$, which is the same as Wentzel's estimate.[3]

In conclusion I express my thanks to Academician N. N. Bogolyubov for suggesting this research and to V. V. Tolmachev for valuable discussions.

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VISCOSITY IN THE HYDRODYNAMIC THEORY OF MULTIPLE PARTICLE PRODUCTION

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The model of a viscous ultrarelativistic fluid is used to describe the dispersion of the meson-nucleon cloud produced in the collision of high-energy nucleons. An asymptotic solution of one-dimensional equations is obtained. It is shown that when viscosity is taken into account the angular distribution of secondary particles is less anisotropic than in the case of an ideal fluid.

1. In Landau's hydrodynamic theory[1,2] of multiple particle production in the collisions of high-energy nucleons the expansion of the meson-nucleon cloud is described as the dispersion of an ultrarelativistic ideal fluid.

Although the theory agrees satisfactorily with experiment in general there are experimental indications that the angular distribution of secondary particles is less anisotropic than is predicted by Landau's theory.[3] It is therefore of interest to investigate the expansion of the system using the model of an ultrarelativistic viscous fluid. The energy dissipation which occurs during the motion of a viscous substance is accompanied by increased entropy of the system. Therefore, in distinction from Landau's theory, additional particles are produced in the expanding meson cloud.

The study of the dissipative processes from the hydrodynamic point of view leads to new parameters — the phenomenological coefficients of viscosity, thermal conductivity etc. The kinematic coefficients could theoretically be calculated from a specific type of interaction in the meson-nucleon cloud; this would involve quantum-statistical averaging of the field operators and a hydrodynamic description of the interacting particles (see reference 4, for example). However, since we do not know the real form of the interaction in the meson cloud at such high energies the indicated procedure is extremely tentative. In the present paper we therefore consider only the kinematic aspect of the problem; we assume the viscosity coefficient to have a given constant value independent of temperature and determine its influence on the characteristics of an elementary act. The viscosity coefficient could be determined experimentally by comparing the theory with experimental data. Two papers of Hamaguchi[5,6] are concerned with the role of viscosity in the hydrodynamic theory of multiple particle production and will be analyzed below.

2. The equations of relativistic hydrodynamics which take dissipative processes (viscosity and thermal conduction) into account are as follows:[7]

$$\partial T_i^k / \partial x^k = 0, \qquad \partial n^i / \partial x^i = 0, \tag{1}$$

$$T_{ik} = wu_iu_k + pg_{ik} + \tau_{ik}, \qquad n_i = nu_i + \nu_i. \tag{2}$$

Here u_i is the velocity 4-vector, which satisfies the relation $u^iu_i = -1$; p is pressure; $w = p + \epsilon$ is the enthalpy density; ϵ is the energy density; n is the particle density; n_i is the particle current vector; g_{ik} is the metric tensor ($g_{11} = g_{22} = g_{33} = 1$, $g_{00} = -1$);

$$\tau_{ik} = -\zeta_1\left(\frac{\partial u_i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} + u_iu^l\frac{\partial u_k}{\partial x^l} + u_ku^l\frac{\partial u_i}{\partial x^l}\right) - \left(\zeta_2 - \frac{2}{3}\zeta_1\right)\frac{\partial u^l}{\partial x^l}(u_iu_k + g_{ik}); \tag{3}$$

$$\nu_i = -\frac{\varkappa}{c}\left(\frac{nT}{w}\right)^2\left[\frac{\partial}{\partial x^i}\left(\frac{\mu}{T}\right) + u_iu_k\frac{\partial}{\partial x^k}\left(\frac{\mu}{T}\right)\right]. \tag{4}$$

ζ_1 and ζ_2 are the two (positive) viscosity coefficients; c is the velocity of light (we shall let $c = 1$); T is the temperature; μ is the chemical potential; κ is the thermal conductivity. The viscosity tensor satisfies the relation[7]

$$\tau_k^i u^k = 0. \tag{5}$$

For the expanding meson-nucleon cloud we assume with Landau that $\mu = 0$. Since then $\nu_i = 0$ thermal conductivity is absent from the Fermi-Landau system. Assuming furthermore $\epsilon = 3p$ for the equation of state, we obtain the equation[5]

$$\tau_i^i = 0. \tag{6}$$

for the tensor τ_{ik}. Equation (6) will always be satisfied if $\zeta_2 = 0$, which we shall assume hereinafter.

We now consider the one-dimensional symmetrical expansion of an infinite plane layer of thickness Δ in a vaccum with viscosity taken into account. The equations in (1) are now represented more conveniently by[3]

$$\frac{1}{3}\frac{\partial\varepsilon}{\partial x^i}+\frac{4}{3}\varepsilon\left[u^k\frac{\partial u_i}{\partial x^k}-\frac{1}{3}u_i\frac{\partial u^k}{\partial x^k}\right]$$
$$=-\frac{16}{9}\zeta u_i\left(\frac{\partial u^l}{\partial x^l}\right)^2-\frac{\partial\tau_i^k}{\partial x^k}. \quad (7)$$

We transform to the new independent variables t and z by means of

$$x^0=t\cosh z,\qquad x^1=t\sinh z. \quad (8)$$

This is equivalent to a transition to a system of curvilinear coordinates in which the matter is almost at rest. Milekhin[3] has made successful use of this transformation in investigating the motion of an ultrarelativistic ideal fluid. Equation (7) is now written for arbitrary curvilinear coordinates through replacement of the ordinary derivatives by covariant derivatives:[3,7]

$$\frac{1}{3}\frac{\partial\varepsilon}{\partial x^i}+\frac{4}{3}\varepsilon\left[u^k\left(\frac{\partial u_i}{\partial x^k}-\frac{1}{2}u^l\frac{\partial g_{kl}}{\partial x^i}\right)\right.$$
$$\left.-\frac{1}{3}\frac{u_i}{\sqrt{-g}}\frac{\partial(\sqrt{-g}u^k)}{\partial x^k}\right]=-\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\tau_i^k)}{\partial x^k}$$
$$+\frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\tau^{kl}-\frac{16}{9}\zeta u_i\left[\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}u^l)}{\partial x^l}\right]^2. \quad (9)$$

The tensor τ_{ik} must also be given in covariant form.

We assume that in the original coordinate system we have $u^0=\cosh\eta$, $u'=\sinh\eta$. In the system represented by (8) the components of the 4-velocity and metric tensor are[3]

$$u^0=\cosh(\eta-z),\quad u^1=t^{-1}\sinh(\eta-z),$$
$$g^{00}=-1,\quad g^{11}=t^{-2},$$
$$g^{22}=g^{33}=1,\quad g^{ik}=0\ \text{for}\ i\neq k. \quad (10)$$

We assume furthermore that $\eta'=\eta-z$ (η' will be denoted by η for convenience). The components of τ_{ik} in the variables t, z using (10), are

$$\tau_0^0=\frac{4}{3}\zeta\sinh^2\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right],$$
$$\tau_1^1=-\frac{4}{3}\zeta\cosh^2\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right],$$
$$\tau_0^1=\frac{4}{3}\zeta\sinh\eta\cosh\eta\left[\frac{\cosh\eta}{t^2}\left(1+\frac{\partial\eta}{\partial z}\right)+\frac{\sinh\eta}{t}\frac{\partial\eta}{\partial t}\right],$$
$$\tau_1^0=-t^2\tau_0^1. \quad (11)$$

After simple but laborious calculations (9) becomes

$$\frac{1}{3}\frac{\partial\varepsilon}{\partial t}-\frac{4}{9}\left[\sinh2\eta\frac{\partial\eta}{\partial t}+\frac{1}{t}(\cosh2\eta-2)\left(1+\frac{\partial\eta}{\partial z}\right)\right]\varepsilon$$
$$=\frac{16}{9}\zeta\cosh\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right]^2$$
$$-\frac{\partial\tau_0^0}{\partial t}-\frac{\tau_0^0}{t}-\frac{\partial\tau_0^1}{\partial z}+t\tau^{11},$$
$$\frac{1}{3}\frac{\partial\varepsilon}{\partial z}+\frac{4}{9}\left[t(\cosh2\eta+2)\frac{\partial\eta}{\partial t}+\sinh2\eta\left(1+\frac{\partial\eta}{\partial z}\right)\right]\varepsilon$$
$$=-\frac{16}{9}\zeta t\sinh\eta\left[\sinh\eta\frac{\partial\eta}{\partial t}+\frac{\cosh\eta}{t}\left(1+\frac{\partial\eta}{\partial z}\right)\right]^2$$
$$-\frac{\partial\tau_1^0}{\partial t}-\frac{\tau_1^0}{t}-\frac{\partial\tau_1^1}{\partial z}. \quad (12)$$

The solution of (12) will be sought for $t\gg\Delta$. We shall also assume $\partial\eta/\partial z\ll1$, $\partial\eta/\partial t<\eta/t$ for $\eta\ll1$, which will be justified by the subsequent solution. Equation (12) can then be put into the simple form

$$\frac{3}{4}\frac{\partial\varepsilon}{\partial\tau}+\varepsilon=\frac{\zeta}{\Delta}e^{-\tau},\qquad\frac{3}{4}\frac{\partial\varepsilon}{\partial z}+2\eta\varepsilon=-\frac{\zeta}{\Delta}\eta e^{-\tau}. \quad (13)$$

Here $\tau=\ln(t/\Delta)$. In (13) $\sinh\eta$ and $\cosh\eta$ are expanded in powers of η but we retain only terms that are linear in η. The second equation in (13) shows that ϵ depends slightly on z. Assuming $z/\tau\ll1$, we therefore obtain a solution for ϵ which satisfies the initial condition $\epsilon\sim\epsilon_0$ for $\tau\sim1$:

$$\varepsilon=\varepsilon_0\left[\exp\left\{-\frac{4}{3}\left(\tau+\frac{z^2}{2\tau}\right)\right\}\right.$$
$$\left.+\gamma e^{-\tau}-\gamma\exp\left\{-\frac{4}{3}\left(\tau-\frac{z^2}{2\tau}\right)\right\}\right],\quad\gamma=4\tau/\varepsilon_0\Delta. \quad (14)$$

It must be noted that (14) is an asymptotic solution which is correct only in order of magnitude for $\tau\sim1$ and $z\sim\tau$.

The second equation of (13) now gives

$$\eta\approx\frac{z}{\tau}\frac{1+\gamma\exp(4z^2/3\tau)}{2+\frac{9}{4}\gamma\exp\{(\tau+2z^2/\tau)/3\}-2\gamma\exp(4z^2/3\tau)}. \quad (15)$$

It is thus evident that the assumptions $\partial\eta/\partial z\sim1/\tau$; $\partial\eta/\partial\tau<\eta$ are actually satisfied for $z/\tau\ll1$ and $\tau\gg1$.

We now perform the new transformation of variables

$$\alpha=\tau+z,\qquad\beta=\tau-z;$$
$$\alpha=\ln\frac{x^0+x^1}{\Delta},\qquad\beta=\ln\frac{x^0-x^1}{\Delta}, \quad (16)$$

which is equivalent to transforming to the customary coordinate system used in references 1, 2, 5, and 6. The criterion for the use of our solution in the new variables will be the condition $\alpha\sim\beta$; this corresponds to the region of maximum particle density in the case of an ideal fluid.

The expression for the energy density becomes

$$\varepsilon = \varepsilon_0 \left\{ \exp\left[-\frac{4}{3}(\alpha + \beta - \sqrt{\alpha\beta}) \right] + \gamma \exp\left[-\frac{\alpha+\beta}{2} \right] - \gamma \exp\left[-\frac{4}{3}\sqrt{\alpha\beta} \right] \right\}. \quad (17)$$

The 4-velocity will be approximated by

$$u^2 \approx e^{2\eta}(x^0 + x^1)/2(x^0 - x^1). \quad (18)$$

The first term in (17) is Landau's solution for an ideal fluid; the second and third terms take viscosity into account. It is interesting that in the region $\alpha \sim \beta$ the energy density in the viscous fluid is greater than in the ideal fluid, whereas for the region of the most energetic particles, $\beta \sim 0$, the energy density may be lower than in the ideal case. As a consequence the leading edge of the matter will disintegrate more rapidly (with smaller α for a given value of β), and consequently the fastest particles will possess relatively lower energy, a smaller fraction of the particles will be emitted at small angles and the angular distribution of the produced particles will be less anisotropic.

With very high viscosity, $\gamma \sim 1$, our solution is valid only in the region $\alpha \approx \beta$. For the exact form of secondary-particle angular and energy distributions we would have to solve extremely complicated three-dimensional second order partial differential equations. However, it can be maintained that the nondisintegrating part of the meson-nucleon cloud will be located in the region $\alpha \sim \beta$, and since $u^2 \sim 1$ in this region all secondary particles will possess approximately the same energy and their angular distribution will obviously be isotropic. The number N of mesons produced will be given by the formula of Rozental' and Chernavskiĭ:[8] $N \sim E_c/\mu$, where E_c is the nucleon energy in the center-of-mass system and μ is the mass of a pion.

3. Hamaguchi[5,6] also investigated the influence of viscosity on the principal features of multiple production as described by the hydrodynamic theory. However, his solution of the one-dimensional hydrodynamic equations is incorrect because of a number of mistakes, some of which we shall now discuss.

In reference 5 the sought solution is expanded in powers of the small viscosity coefficient in order to solve the hydrodynamic equations for one-dimensional expansion. The Landau solution is taken as the zeroth approximation. By retaining only linear terms in the viscosity coefficient the author obtains a set of linear partial differential equations with variable coefficients as a basis for the given approximation. For the energy density ϵ_0 in the solution which neglects viscosity he assumes, as a quite rough approximation of Landau's solution, $\epsilon_0 = \epsilon_0^* \exp[-4/3(\alpha - \beta)]$. In addition to the fact that for the region of maximum particle density $\alpha \sim \beta$ the expression for ϵ_0 is incorrect, it must be noted that this approximation seriously neglects terms which are of the same order of magnitude as other terms left in the equations. The original equations contain the terms $\partial\epsilon_0/\partial\xi$ and $\partial\epsilon_0/\partial x^0$ (here $\xi = x^0 \quad x'$); it is easily seen that when $\epsilon_0^* \exp[-4/3(\alpha + \beta - \sqrt{\alpha\beta})]$ (Landau's solution) is replaced by ϵ_0 the terms $\frac{1}{2}\sqrt{\beta/\alpha}$ and $\frac{1}{2}\sqrt{\alpha/\beta}$ are neglected compared with unity although both of these terms are of the order of unity for the important region $\alpha - \beta$.

For the correction of 4-velocity u_1 Hamaguchi seeks a solution in the form $u_1^2 = f_1(x^0/\xi)^N$, where N is a number selected to make f_1 slightly dependent on x^0 and ξ in the solution. However, [see Eq. (24) of reference 5], f_1 is found to be always strongly dependent on its arguments. It can therefore be shown that any number greater than 4 can be taken for N instead of the value 4.541 given by Hamaguchi. Since the value of N strongly affects the solutions the uncertainty of its value makes the solution somewhat arbitrary.

In conclusion the author wishes to thank I. L. Rozental' for suggestions and assistance, and G. A. Milekhin and D. S. Chernavskiĭ for valuable discussions.

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ROTATIONAL STATES OF ODD NUCLEI WITHOUT AXIAL SYMMETRY

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A theory is developed for the rotational states of odd nuclei whose ground state spin is due to the angular momentum $j = \frac{1}{2}$ of the outer electron. The energy of the rotational states is obtained as a function of the parameter γ $(0 \leq \gamma \leq \pi/3)$ which determines the deviation of the nuclear shape from axial symmetry.

INTRODUCTION

It was shown by Davydov and Filippov[1] that many properties of the first excited states of even-even nuclei (the order of succession of the spins of the excited states, their energies, and the probabilities for electromagnetic transitions between them) can be readily explained by assuming that the equilibrium shape of the nucleus can in first approximation be represented by a three-axial ellipsoid, which is characterized by two parameters, β and γ (see the work of Bohr[2]). The relations

$$a_0 = \beta\cos\gamma, \qquad a_1 = a_{-1} = 0, \qquad a_2 = a_{-2} = (\beta/\sqrt{2})\sin\gamma$$

connect the parameters β and γ with the quantities a_μ which determine the shape of the nucleus:

$$R(\vartheta, \varphi) = R_0 + R_0 \sum_{\mu=-2}^{2} a_\mu Y_{2\mu}(\vartheta, \varphi)$$

in a coordinate system attached to the nucleus. Changing the "asymmetry" parameter γ from 0 to $\pi/3$, with a fixed value β, induces a change of the nuclear shape from a prolate to an oblate ellipsoid of revolution. The value $\gamma = 30°$ corresponds to an intermediate shape of the nucleus between the prolate and the oblate ellipsoids of revolution.

The possibility of departing from the axial symmetry of the equilibrium shape of the nucleus has been investigated in the papers of Geilikman,[3] Zaikin,[4] and Davydov and Fillippov.[5] In the present paper we consider the rotational states of odd nuclei under the assumption that the shape of the nucleus is determined by the fixed equilibrium values of the parameters β and γ, with the odd nucleon in a state with the definite total angular momentum $j = \frac{1}{2}$.

In keeping with the fact that we wish to consider the general case with arbitrary values of the parameter γ, we shall not assume that the projection of the total angular momentum $j_3 = \Omega$ is an integral of the motion. Instead, we seek exact solutions of the Schrödinger equation giving the rotational energy of the nucleus without neglecting (as was done in the papers of Bohr,[2] Bohr and Mottelson,[6] Davidson and Feenberg,[7] and others) the nondiagonal matrix elements connected with the different values of Ω.

In section 1 we obtain the energies of the rotational states of the nucleus for fixed parameters β and γ. We show, in particular, that for $\gamma = 0$ the exact solutions coincide with the solutions obtained in the aforementioned papers with neglect of the nondiagonal matrix elements of the operator of the rotational energy of the nucleus. In Sec. 2 we compare the theory with the experimental data for the nucleus W^{183}.

1. ROTATIONAL ENERGY LEVELS OF ODD NUCLEI WITH THE OUTER NUCLEON IN THE STATE $j = \frac{1}{2}$

We consider, as a model of the nucleus, a system consisting of a single outer nucleon in the state $j = \frac{1}{2}$ and a nuclear core with the shape of a three-axial ellipsoid. In other words, we assume that the core of the nucleus consists of nucleons belonging to filled shells as well as of "paired" nucleons of the outer shell, which are responsible for the deviation of the shape of the nuclear core from spherical symmetry.

If I_λ are the projections of the total angular momentum operator on the three principal axes of the ellipsoid describing the shape of the nucleus, and j_λ the corresponding projections of the total angular momentum of the outer nucleon, then the nuclear Hamiltonian which conserves the total angular momentum of the nucleon can, in the adiabatic approximation (neglecting the operator of the kinetic energy corresponding to changes of β and γ) be written in the form proposed by Bohr:[2]

$$H = \hbar^2 (8B\beta^2)^{-1} \sum_{\lambda} \frac{(\hat{I}_\lambda - \hat{j}_\lambda)^2}{\sin^2(\gamma - 2\pi\lambda/3)} + H_p + H_{int},$$

Here the first term is the rotational Hamiltonian; H_p is the Hamiltonian of the outer nucleon;

$$H_{int} = T\beta \{\cos\gamma\,(3\hat{j}_3^2 - \hat{j}^2) + \sqrt{3}\sin\gamma\,(\hat{j}_1^2 - \hat{j}_2^2)\}$$

is the Hamiltonian describing the interaction of the outer nucleon with the deformation of the nuclear surface.

In the states with $j = \frac{1}{2}$, $H_{int} = 0$, and the rotational energy of the nucleus (without changes in the internal state) can be obtained from the eigenvalues of the operator

$$H_r = \hbar^2 (8B\beta^2)^{-1} \sum_{\lambda} (I_\lambda - j_\lambda)^2 / \sin^2\left(\gamma - \frac{2\pi}{3}\lambda\right). \quad (1.1)$$

The wave function for the rotational state of the odd nucleus satisfying the symmetry conditions given by Bohr[2] (invariance of the nuclear shape under rotations of 180° about each of the axes 1, 2, and 3) and corresponding to the total angular momentum I of the nucleus can be written in the form

$$\psi_I = \sum_{K,\Omega} |\, I\, jK\Omega \rangle\, A_{K\Omega}, \quad (1.2)$$

where

$$|\, IjK\Omega \rangle = \left(\frac{2I+1}{16\pi^2}\right)^{1/2} \{\varphi_\Omega D^I_{MK} + (-1)^{I-j}\varphi_{-\Omega} D^I_{M,-K}\}, \quad (1.3)$$

with

$$K - \Omega = 2\nu, \qquad \nu = 0, \pm 1, \pm 2, \pm \ldots \quad (1.4)$$

The wave function φ_Ω in (1.3) determines the state of the outer nucleon with total angular momentum j and projection Ω; D^I_{MK} are irreducible representations of the three-dimensional rotation group, which depend on the Eulerian angles specifying the orientation of the principal axes in space; K is the projection of the total angular momentum on the third axis of the nucleus, and M is the projection of the total angular momentum on the z axis of the fixed system of coordinates.

Using the known values of the matrix elements of the operator (1.1) with respect to the wave functions φ_Ω and of the quantities D^I_{MK}, we can transform the Schrödinger equation

$$\{H_r - E(I)\}\,\psi_I = 0$$

to a system of linear, homogeneous, algebraic equations for the unknown coefficients $A_{K\Omega}$. The rotational energy E(I) corresponding to the total angular momentum I of the nucleus is given by the secular equation for this system of equations.

For $I = \frac{1}{2}$ we conclude from (1.4) that there exists only one state with $K = \Omega = \frac{1}{2}$. The wave function for this state,

$$\psi_{1/2} = (8\pi^2)^{-1/2} \{\varphi_{1/2} D^{1/2}_{M,1/2} + \varphi_{-1/2} D^{1/2}_{M,-1/2}\} \quad (1.5)$$

satisfies the equation

$$[H_r - E(1/2)]\,\psi_{1/2} = 0,$$

with $E(\frac{1}{2}) = 0$. The function (1.5) therefore corresponds to the ground state of the nucleus. The spin of the ground state is $\frac{1}{2}$.

For $I = 3/2$ formula (1.4) gives two possibilities, $K = \Omega = \frac{1}{2}$ and $\Omega = \frac{1}{2}$, $K = -3/2$. The wave function (1.2) will then be a linear combination of the two functions

$$|\, 3/2, 1/2, 1/2, 1/2 \rangle = (2\pi)^{-1} \{\varphi_{1/2} D^{3/2}_{M,1/2} - \varphi_{-1/2} D^{3/2}_{M,-1/2}\},$$

$$|\, 3/2, 1/2, -3/2, 1/2 \rangle = (2\pi)^{-1} \{\varphi_{1/2} D^{3/2}_{M,-3/2} - \varphi_{-1/2} D^{3/2}_{M,3/2}\}.$$

The energy of the two rotational states belonging to the spin $3/2$, measured in the units $\hbar^2/B\beta^2$, is given by the formula

$$\varepsilon_{1,2}(3/2) = 9\,(1 \mp \sqrt{1 - 8/9 \sin^2 3\gamma})\,/\,4\sin^2 3\gamma. \quad (1.6)$$

For $I = 5/2$ the following values are possible

$$K = \Omega = 1/2; \quad \Omega = 1/2, \quad K = -3/2; \quad \Omega = 1/2, \quad K = 5/2$$

and the wave function (1.2) is then a linear combination of the three functions

$$|\, 5/2, 1/2, 1/2, 1/2 \rangle = (3/8\pi^2)^{1/2} \{\varphi_{1/2} D^{5/2}_{M,1/2} + \varphi_{-1/2} D^{5/2}_{M,-1/2}\},$$

$$|\, 5/2, 1/2, -3/2, 1/2 \rangle = (3/8\pi^2)^{1/2} \{\varphi_{1/2} D^{5/2}_{M,-3/2} + \varphi_{-1/2} D^{5/2}_{M,3/2}\},$$

$$|\, 5/2, 1/2, 5/2, 1/2 \rangle = (3/8\pi^2)^{1/2} \{\varphi_{1/2} D^{5/2}_{M,5/2} + \varphi_{-1/2} D^{5/2}_{M,-5/2}\}.$$

The energy of the three rotational levels with spin $I = 5/2$ is determined by the cubic equation

$$\varepsilon^3 - \frac{9\varepsilon^2}{\sin^2 3\gamma} + \frac{9\,(9 + 2\sin^2 3\gamma)\,\varepsilon}{4\sin^4 3\gamma} - \frac{3\,(27 + \sin^2 3\gamma)}{4\sin^4 3\gamma} = 0.$$

The solutions to this equations for several values γ are listed in Table I.

In the same fashion one can show that the ener-

TABLE I

γ°	0	5	10	15	20	25	30
$\varepsilon_1(5/2)$	1.000	1.018	1.074	1.175	1.331	1.547	1.708
$\varepsilon_2(5/2)$	∞	65.91	16.78	7.682	4.525	3.131	2.646
$\varepsilon_3(5/2)$	∞	67.20	18.10	9.143	6.144	4.968	4.644

TABLE II

$\gamma°$	0	5	10	15	20	25	30
$\varepsilon_1(^7/_2)$	3.333	3.384	3.531	3.750	3.954	4.011	4.0
$\varepsilon_2(^7/_2)$	∞	67.17	18.10	9.10	6.10	4.923	4.5
$\varepsilon_3(^7/_2)$	∞	68.54	19.42	10.62	8.066	7.802	8.5
$\varepsilon_4(^7/_2)$		263.9	67.05	30.63	17.98	12.30	10.0
$\varepsilon_1(^9/_2)$	3.333	3.382	3.531	3.750	3.954	4.000	4.0
$\varepsilon_2(^9/_2)$	∞	68.52	19.43	10.63	8.070	7.800	8.5
$\varepsilon_3(^9/_2)$	∞	70.25	21.19	12.42	9.757	9.23	9.0
$\varepsilon_4(^9/_2)$	∞	263.6	66.92	30.64	18.00	12.30	10.0
$\varepsilon_5(^9/_2)$	∞	266.0	68.93	32.55	20.0	15.0	13.5

gies of the levels with spins $^7/_2$ and $^9/_2$ are given by the solutions of equations of the fourth and fifth degree in ϵ, respectively. The solutions to these equations for several values γ are listed in Table II.

It follows from (1.6) and Tables I and II that for $\gamma = 0$ (axially symmetric nucleus) the energies of the rotational levels coincide (up to an additive constant)* with the energy values given by the formula (in our units $\hbar^2/B\beta^2$)

$$\varepsilon_r = {}^1/_6\{I(I+1) - (-1)^{I-1/2}(I+{}^1/_2) + {}^3/_4\},$$

of Bohr and Mottelson (reference 6, p 22) for an axially symmetric nucleus, neglecting the nondiagonal matrix elements and assuming that $\Omega = K = \frac{1}{2}$.

The energies of the levels with spins $^3/_2$ and $^5/_2$, as well as those with spins $^7/_2$ and $^9/_2$, coincide for an axially symmetric nucleus. As γ (i.e., the deviation from axial symmetry) increases, the energies of the levels of the axially symmetric nucleus are raised somewhat, while at the same time the remaining rotational energy levels with the same spin value come down from infinity. For $\gamma > 20°$ the energy spectrum becomes rather complicated. Several levels with the same spin value correspond to one and the same internal state. Before, these states belonged to different rotational bands.

Figure 1 illustrates how the rotational states behave as γ changes in the interval 10 to 30°.

In Tables I and II we list the values of the rotational energy for various values γ in the interval 0, $\pi/6$. The values of the rotational energy for values of γ in the interval $\pi/6$, $\pi/3$ can be easily obtained by noting that the equations (1.6) to (1.9) are invariant under the transformation $\gamma \to \gamma' = \pi/3 - \gamma$.

We therefore have the equation

$$\varepsilon(I, \gamma) = \varepsilon(I, \pi/3 - \gamma).$$

2. COMPARISON WITH EXPERIMENT

Since we are employing a very idealized model (a core plus one outer nucleon in the state $j = \frac{1}{2}$),

*The rotational energies of the nucleus in the work of Bohr and Mottelson are not normalized to zero for the ground state.

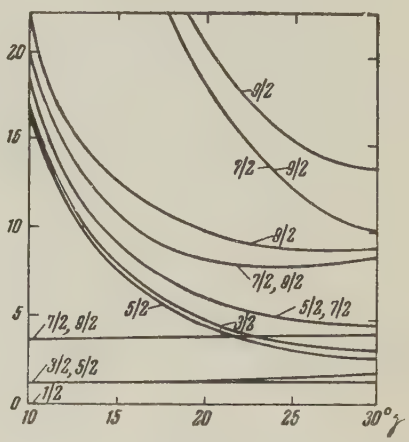

FIG. 1

the results obtained can be applied only to a few nuclei with the spin $\frac{1}{2}$ in the ground state. The deformed nucleus W^{183} apparently is such a nucleus. The right hand side of Fig. 2 represents the level scheme for this nucleus.[8] We indicate the spins and (in parentheses) the ratio of each level energy of the first excited state of the nucleus (46.5 kev). The left hand side of Fig. 2 represents the theoretical level scheme calculated on the basis of the results of the present paper with $\gamma = 27°$. It is seen from the figure that the theory reproduces satisfactorily the energy levels and the corresponding spins. To get a fuller understanding of the agreement with theory, we shall calculate in a future paper the probabilities for electromagnetic transitions between the rotational levels. Furthermore,

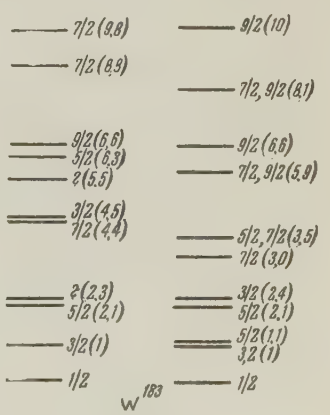

FIG. 2

we shall extend the theory to the case where the outer nucleon is in states with an angular momentum $j \neq 0$.

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ON THE QUANTIZATION OF HALF-INTEGER SPIN FIELDS

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A method of quantization for half-integer spin fields is considered which is different from the usual one involving anticommutators and is consistent with the principle of relativistic causality, positive definiteness of the energy (for non-interacting fields), the Lagrangian formalism in Schwinger's formulation,[1] and with invariance under TCP transformations.[2] The main difference between the proposed method and the usual one is that the maximal occupation number is two.

1. INTRODUCTION

IT is a well-known fact that nonrelativistic quantum mechanics does not explain the connection between spin and statistics. Moreover, the equations of nonrelativistic quantum mechanics admit of solutions which are neither completely symmetric nor completely antisymmetric, and which transform according to different irreducible representations of the permutation group if the particles are interchanged. In the relativistic quantum theory, the existing methods of quantization lead to a unique connection between spin and statistics;[3] however, from the very beginning only two alternatives are considered in this case: either we quantize with commutators, or with anticommutators. In this connection it is of interest to investigate whether other possibilities, which are admissible in nonrelativistic quantum mechanics, are consistent with the basic principles of the relativistic theory.

In the present paper we consider, on the simplest example of the Dirac equation, the possibility of constructing an algebra of operator wave functions with the following properties: it leads to a new statistics with the maximal occupation number two for each individual state,* and is at the same time consistent with the principle of relativistic causality, the positive definiteness of the energy (for non-interacting fields), and with the Lagrangian formalism. In setting up the Lagrangian formalism we make use of the variational principle of Schwinger.[1] We show that this method of quantization is a consequence of the variational principle of Schwinger based on a class of admissible variations of the operator wave function which is different from that used in the usual quantization scheme.

*This method of quantization can be generalized for the case of arbitrary maximal occupational numbers.

2. THE CONDITION OF RELATIVISTIC CAUSALITY

The requirement of relativistic causality is usually formulated as the condition that commutators of physical operators reduce to zero for points which are separated by a space-like interval (outside the light cone). With the condition that operators corresponding to measurable quantities are bilinear combinations of the operator wave functions (as in the case of half-integer spin fields), the requirement of relativistic causality will be fulfilled if the commutators or anticommutators of the operator wave functions reduce to zero outside the light cone. These are the only two cases usually considered. It is, however, possible to satisfy the requirement of relativistic causality using a different algebra for the operator wave functions.

As an example, we consider the field satisfying the free Dirac equation,

$$(\gamma_\mu \partial/\partial x_\mu + m)\psi = 0, \tag{1}$$

where γ_μ ($\mu = 1, 2, 3, 4$) are matrices defined by the relation $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$.

We define the operator properties of the wave functions $\psi(x)$ and $\bar{\psi}(x) = \psi^*(x)\gamma_4$ with the help of the commutation relations

$$\psi_\alpha(x)\psi_\beta(x')\psi_\gamma(x'') + \psi_\gamma(x'')\psi_\beta(x')\psi_\alpha(x) = 0,$$

$$\psi_\alpha(x)\bar{\psi}_\beta(x')\psi_\gamma(x'') + \psi_\gamma(x'')\bar{\psi}_\beta(x')\psi_\alpha(x) = -iS_{\alpha\beta}(x-x')\psi_\gamma(x'') - iS_{\gamma\beta}(x''-x')\psi_\alpha(x),$$

$$\bar{\psi}_\alpha(x)\bar{\psi}_\beta(x')\psi_\gamma(x'') + \psi_\gamma(x'')\bar{\psi}_\beta(x')\bar{\psi}_\alpha(x) = -iS_{\gamma\beta}(x''-x')\bar{\psi}_\alpha(x), \tag{2}$$

where $S(x)$ is the known commutator function[4]

$$S_{\alpha\beta}(x) = -(\gamma_\mu \partial/\partial x_\mu - m)_{\alpha\beta}\Delta(x),$$
$$\Delta(x) = i(2\pi)^{-3}\int e^{ipx}\varepsilon(p)\,\delta(p^2+m^2)\,d^4p. \qquad (3)$$

It follows from the properties of the function $S(x)$ that the relations (2) are consistent with Eq. (1). We verify that all commutators of the form $[\psi_\alpha(x)\psi_\beta(x),\ \psi_\gamma(x')\psi_\delta(x')]$ are zero outside the light cone. For the proof we write the relations (2) in the following compact form:

$$\psi_\alpha(x)\psi_\beta(x')\psi_\gamma(x'') + \psi_\gamma(x'')\psi_\beta(x')\psi_\alpha(x)$$
$$= \{\psi_\alpha(x), \psi_\beta(x')\}_F\psi_\gamma(x'') + \{\psi_\gamma(x''), \psi_\beta(x')\}_F\psi_\alpha(x), \qquad (2')$$

where ψ now stands for the usual as well as the Dirac conjugate spinor, and the brackets with the index F are given by

$$\{\psi_\alpha(x), \bar\psi_\beta(x')\}_F \equiv -iS_{\alpha\beta}(x-x'),$$
$$\{\psi_\alpha(x),\ \psi_\beta(x')\}_F \equiv 0, \qquad (4)$$

i.e., by the usual values for the anticommutators in the quantization according to Fermi-Dirac statistics.*

Using the relations (2'), we can transform this commutator to the form

$$\psi_\alpha(x)\psi_\beta(x)\psi_\gamma(x')\psi_\delta(x') - \psi_\gamma(x')\psi_\delta(x')\psi_\alpha(x)\psi_\beta(x)$$
$$= -\{\psi_\alpha(x), \psi_\delta(x')\}_F\psi_\gamma(x')\psi_\beta(x)$$
$$+ \{\psi_\beta(x), \psi_\gamma(x')\}_F\psi_\alpha(x)\psi_\delta(x').$$

It follows from the properties of the function $S(x)$ that this expression is zero outside the light cone.

It is easily shown that the commutators of physical quantities also reduce to zero outside the light cone for commutation relations of the type

$$\psi_\alpha(x)\psi_\beta(x')\psi_\gamma(x'') + \psi_\gamma(x'')\psi_\beta(x')\psi_\alpha(x)$$
$$= \{\psi_\alpha(x), \psi_\beta(x')\}_F\psi_\gamma(x'') + \{\psi_\gamma(x''), \psi_\beta(x')\}_F\psi_\alpha(x)$$
$$+ \rho\{\psi_\alpha(x), \psi_\gamma(x'')\}_F\psi_\beta(x'); \qquad (5)$$

$$\psi_\alpha(x)\psi_\beta(x')\psi_\gamma(x'') - \psi_\gamma(x'')\psi_\beta(x')\psi_\alpha(x)$$
$$= \{\psi_\alpha(x), \psi_\beta(x')\}_F\psi_\gamma(x'') - \{\psi_\gamma(x''), \psi_\beta(x')\}_F\psi_\alpha(x), \qquad (6)$$

where ρ is an arbitrary number.

The commutation relations (5) have non-zero solutions only for $\rho = 0$ and $\rho = -1$. In the last case the algebra for the operator wave functions corresponds to the usual method of quantization using anticommutators.

The commutation relations (6) differ from the commutation relations (2') by a change of sign. This difference (as in the case of commutators and anticommutators) leads to an energy for half-integer spin fields which is not positive definite.

*We note that in this quantization scheme the anticommutator $\{\psi_\alpha(x), \psi_\beta(x')\} \neq \{\psi_\alpha(x), \psi_\beta(x')\}_F$.

In principle, the relations (6) can be used for the quantization of fields with integer spin, with the requirement that the interaction Lagrangian contains an even number of field operators.

3. MOMENTUM REPRESENTATION

The algebra for the field operators is conveniently realized in the momentum representation. We make the transition to the momentum representation by expanding the operator wave functions and the commutator function $S(x)$ into Fourier series:[4]

$$\psi_\alpha(x) = V^{-1/2}\sum_{\mathbf{p}}\sum_{r=1}^{2}\{a_{\mathbf{p}r}u^r_\alpha(\mathbf{p})e^{ipx} + b^+_{\mathbf{p}r}v^r_\alpha(\mathbf{p})e^{-ipx}\},$$

$$\bar\psi_\alpha(x) = V^{-1/2}\sum_{\mathbf{p}}\sum_{r=1}^{2}\{a^+_{\mathbf{p}r}\bar u^r_\gamma(\mathbf{p})e^{-ipx} + b_{\mathbf{p}r}\bar v^r_\alpha(\mathbf{p})e^{-ipx}\}, \qquad (7)$$

$$-iS_{\alpha\beta}(x) = \frac{1}{V}\sum_{\mathbf{p}}\sum_{r=1}^{2}\{u^r_\alpha(\mathbf{p})\,\bar u^r_\beta(\mathbf{p})\,e^{ipx}$$
$$+ v^r_\alpha(\mathbf{p})\,\bar v^r_\beta(\mathbf{p})\,e^{-ipx}\}, \qquad (8)$$

where V is the normalization volume; the sum over r implies summation over the states with different polarization; u^r_α and v^r_α are constant spinors subject to the orthonormality conditions

$$\sum_\alpha u^{r*}_\alpha u^s_\alpha = \delta_{rs}, \qquad \sum_\alpha v^{r*}_\alpha v^s_\alpha = \delta_{rs}.$$

Substituting the expansions (7) and (8) in (2), we obtain the following commutation relations for the operators a and b:

$$a_k a^+_l a_m + a_m a^+_l a_k = \delta_{kl}a_m + \delta_{ml}a_k,$$
$$a_k a^+_l a^+_m + a^+_m a^+_l a_k = \delta_{kl}a^+_m,$$
$$b_k b^+_l b_m + b_m b^+_l b_k = \delta_{kl}b_m + \delta_{ml}b_k,$$
$$b_k b^+_l b^+_m + b^+_m b^+_l b_k = \delta_{kl}b^+_m,$$
$$a_k a^+_l b_m + b_m a^+_l a_k = \delta_{kl}b_m,$$
$$b_k b^+_l a_m + a_m b^+_l b_k = \delta_{kl}a_m; \qquad (9)$$

the indices k, l, and m define the momentum and the polarization.

All the remaining commutation relations of the same type, except those which derive from (9) by Hermitian conjugation, are equal to zero.*

The operators corresponding to the basic physical quantities can be simply expressed in terms of the operators $N_k = a^+_k a_k - a_k a^+_k + 1$ [cf. Eqs.

*With $a_k = \alpha_k + i\beta_k$ and $b_k = \gamma_k + i\delta_k$, where α_k, β_k, γ_k, and δ_k are Hermitian matrices, relations (9) go over into the Duffin-Kemmer relations. The algebra of the Duffin-Kemmer matrices for an arbitrary number of matrices was considered by Fujiwara.[5]

(17′) to (19′) below]. We show that this operator can be interpreted as the operator corresponding to the number of particles in the state k.

To determine the eigenvalues of the operator N_k, we use the relation

$$(a_k^+ a_k - a_k a_k^+)^3 = a_k^+ a_k - a_k a_k^+, \tag{10}$$

which is readily proven with the help of the commutation relations (9). It follows from formula (10) that the eigenvalues of the operator $a_k^+ a_k - a_k a_k^+$ are equal to -1, 0, or 1, i.e., the corresponding eigenvalues of the operator N_k are equal to 0, 1, or 2.

We consider the commutation relations of the operator N_k with the operators a_l and a_l^+:

$$[a_k^+ a_k - a_k a_k^+, a_l]$$

$$= a_k^+ a_k a_l - a_k a_k^+ a_l - a_l a_k^+ a_k + a_l a_k a_k^+. \tag{11}$$

From the relations (9) we have

$$a_k^+ a_k a_l + a_l a_k a_k^+ = a_l,$$

$$a_k a_k^+ a_l + a_l a_k^+ a_k = a_l + \delta_{lk} a_k. \tag{12}$$

Substituting (12) in (11), we obtain

$$[N_k, a_l] = -\delta_{kl} a_l. \tag{13}$$

Similarly, we have for the operator a_l^+:

$$[N_k, a_l^+] = \delta_{kl} a_l^+. \tag{14}$$

The relations (13) and (14) are analogous to the corresponding relations in the usual quantization scheme. In particular, they imply that the operators a_l^+ and a_l can be interpreted as the creation and annihilation operators for particles in the state l.

Indeed, if $\Psi_{n_1,n_2,\ldots}$ is the simultaneous eigenvector of the operators N_k (the operators N_k commute with one another) with the eigenvalues n_k, then $a_k \Psi_{n_1,\ldots}$ and $a_k^+ \Psi_{n_1,\ldots}$ are also eigenvectors of these operators, where the eigenvalue n_k is lowered or raised by unity, respectively.

We define the vacuum state as the state in which all occupation numbers n_k are equal to zero, i.e.,

$$(a_k^+ a_k - a_k a_k^+ + 1)\Psi_0 = 0. \tag{15}$$

It follows from the definition (15) and the relations (13) and (14) that

$$a_k \Psi_0 = 0, \quad a_k a_l^+ \Psi_0 = 0 \quad (k \neq l);$$

$$a_k a_k^+ \Psi_0 = \Psi_0. \tag{16}$$

We can generate a complete set of basis vectors by successively acting on the vacuum vector with the creation operators a_k^+:

$$\Psi_0, \; a_k^+ \Psi_0, \qquad a_k^+ a_l^+ \Psi_0 \text{ etc.}$$

We note that, in contrast to the quantization scheme using anticommutators, we now have $a_k^+ a_k^+ \Psi_0 \neq 0$; furthermore, basis vectors differing in the order of the operators a_k^+ can be independent. For example, in the case of two particles the vectors $a_k^+ a_l^+ \Psi_0$ and $a_l^+ a_k^+ \Psi_0$ are independent; for three particles, we have the following independent vectors:

$$a_k^+ a_l^+ a_m^+ \Psi_0, \quad a_k^+ a_m^+ a_l^+ \Psi_0, \quad a_m^+ a_k^+ a_l^+ \Psi_0$$

etc. This difference manifests itself in configuration space through the appearance of partially symmetric wave functions.

The commutation relations (9) together with the relations (16) permit the calculation of the result of operating with the operators a_k and a_k^+ on an arbitrary basis vector; thus we can determine the explicit form of these operators in the representation under consideration.

4. OPERATORS CORRESPONDING TO PHYSICAL QUANTITIES

The operators of energy, momentum, and charge for the free Dirac field are given by the following expressions, which are antisymmetric in the operator wave functions:

$$E = i\int \left(\bar{\psi}\gamma_4 \frac{\partial}{\partial t}\psi - \frac{\partial}{\partial t}\psi\gamma_4^T\bar{\psi}\right) dV; \tag{17}$$

$$\mathbf{P} = -i\int (\bar{\psi}\gamma_4 \nabla\psi - \nabla\psi\gamma_4^T\bar{\psi})\, dV; \tag{18}$$

$$Q = e\int (\bar{\psi}\gamma_4\psi - \psi\gamma_4^T\bar{\psi})\, dV. \tag{19}$$

The conventional expression for the energy, momentum, and charge differ from the expressions (17) to (19) by the factor $\frac{1}{2}$. This difference is connected with the normalizations of the commutation relations for the operator wave functions; it can be removed by changing the function $S(x-x')$ in the commutation relations to $2S(x-x')$.

In the momentum representation the operators of energy, momentum, and charge have the form

$$E = \sum_{pr} |p_0| (a_{pr}^+ a_{pr} - a_{pr} a_{pr}^+ - b_{pr} b_{pr}^+ + b_{pr}^+ b_{pr})$$

$$= \sum_{pr} |p_0| (N_{pr}^{(+)} + N_{pr}^{(-)} - 2); \tag{17′}$$

$$\mathbf{P} = \sum_{pr} \mathbf{p} (a_{pr}^+ a_{pr} - a_{pr} a_{pr}^+ - b_{pr} b_{pr}^+ + b_{pr}^+ b_{pr})$$

$$= \sum \mathbf{p} (N_{pr}^{(+)} + N_{pr}^{(-)}); \tag{18′}$$

$$Q = e\sum (a_{pr}^+ a_{pr} - a_{pr} a_{pr}^+ + b_{pr} b_{pr}^+ - b_{pr}^+ b_{pr})$$

$$= e \sum_{pr} (N_{pr}^{(+)} - N_{pr}^{(-)}), \tag{19′}$$

where $N_{pr}^{(+)}$ and $N_{pr}^{(-)}$ are the number operators for particles and antiparticles.

The infinite term $\sum 2|p_0|$ in expression (17) does not contain any operators and can therefore be omitted, as in the usual theory. As a result, the energy becomes a positive definite quantity. The spectrum of the operators E, **P**, and Q admits of the usual interpretation in terms of the number of particles occupying the individual states, with the only difference that now the maximal occupation number for each state is equal to two.

We now consider the commutators of operators for physical quantities with the operator wave functions. From the relations (13) and (14) we have

$$[E, \psi(x)] = -i\partial\psi(x)/\partial t,$$
$$[\mathbf{P}, \psi(x)] = i\nabla\psi(x),$$
$$[Q, \psi(x)] = -\psi(x). \tag{20}$$

The relations (20) give the usual connection between the operators of energy, momentum, and charge and the infinitesimal canonical transformations. Similar relations can also be obtained for other physical operators.

5. THE VARIATIONAL PRINCIPLE OF SCHWINGER. TCP INVARIANCE

The variational principle of Schwinger[1] contains the most consistent formulation of the basic postulates of the quantum theory of localized fields. We show that our method of quantization is contained in the variational principle of Schwinger as a special solution.

Since the detailed exposition of Schwinger's variational principle can be found in the literature, we shall deal only with those aspects which change as we make the transition to our method of quantization.

We take the Lagrangian in the form

$$L = -\frac{1}{2}\left[\bar{\psi}, \gamma_\mu \frac{\partial}{\partial x_\mu}\psi + m\psi\right]$$
$$-\frac{1}{2}\left[-\frac{\partial}{\partial x_\mu}\bar{\psi}\gamma_\mu + m\bar{\psi}, \psi\right] + \ldots, \tag{21}$$

.... stands for any arbitrary interaction terms.

We shall assume that the terms describing the interaction are antisymmetrized with respect to the operator functions ψ and $\bar{\psi}$. The class of admissible variations is restricted by the conditions

$$\psi_\alpha(x)\,\delta\psi_\beta(x)\,\psi_\gamma(x) + \psi_\gamma(x)\,\delta\psi_\beta(x)\,\psi_\alpha(x) = 0; \tag{22}$$

$$\delta\psi_\alpha(x)\,(\psi_\beta(x)\psi_\gamma(x) - \psi_\gamma(x)\psi_\beta(x))$$
$$+ (\psi_\gamma(x)\psi_\beta(x) - \psi_\beta(x)\psi_\gamma(x))\,\delta\psi_\alpha(x) = 0. \tag{23}$$

where $\psi(x)$ stands for the usual as well as the Dirac conjugate spinor. We shall see later on that we have to supplement the definition of the class of admissible variations to obtain the commutation relations.

Condition (23) is sufficient for the derivation of the equations of motion. Indeed, owing to the antisymmetry of the Lagrangian, condition (23) permits us to move the variations either completely to the right or to the left, depending on whether the variation is in an even or an odd position in the formula. Again we see from the antisymmetry of the Lagrangian that the coefficients of the variations standing to the left or to the right in the formula are equal, and can be set equal to zero simultaneously.

To obtain the commutation relations, we consider the operators $G(\psi)$ and $G(\bar{\psi})$ which generate an infinitesimal transformation of the functions $\psi(x)$ and $\bar{\psi}(x)$;[1]

$$G(\psi) = i\int dV\,[\bar{\psi}(x), \gamma_4\delta\psi(x)]; \tag{24}$$

$$G(\bar{\psi}) = -i\int dV\,[\delta\bar{\psi}(x), \gamma_4\psi(x)]; \tag{25}$$

the time t is assumed to be the same in both operators and is not indicated explicitly.

The commutators of G with ψ and $\bar{\psi}$ are equal to

$$[\psi(x), G(\psi)] = i\delta\psi(x); \tag{26}$$

$$[\bar{\psi}(x), G(\psi)] = i\delta\psi(x). \tag{27}$$

The other commutators are equal to zero.

Substituting (24) in (26) and using (22), we find

$$\int dV\,\{\psi_\mu(x')\,\bar{\psi}_\nu(x)\,(\gamma_4)_{\nu\rho}\,\delta\psi_\rho(x)$$
$$+ \delta\psi_\rho(x)\,(\gamma_4)_{\nu\rho}\,\bar{\psi}_\nu(x)\,\psi_\mu(x')\} = \delta\psi_\mu(x'). \tag{28}$$

Hence

$$\psi_\mu(x')\,\bar{\psi}_\nu(x)\,(\gamma_4)_{\nu\rho}\delta\psi_\rho(x) + \delta\psi_\rho(x)\,(\gamma_4)_{\nu\rho}\,\bar{\psi}_\nu(x)\,\psi_\mu(x')$$
$$= \delta(x'-x)\,\delta\psi_\mu(x). \tag{29}$$

Analogous relations are obtained for the other commutators of G with ψ and $\bar{\psi}$.

The relations (29) further delimits the class of admissible variations. However, this delimitation is not sufficiently complete to obtain the commutation relations in explicit form.

We note that the more general relations of the type

$$\psi_\mu(x)\,\bar{\psi}_\nu(x')\,\delta\psi_\rho(x'') + \delta\psi_\rho(x'')\,\bar{\psi}_\nu(x')\,\psi_\mu(x)$$
$$= (\gamma_4)_{\mu\nu}\,\delta(x-x')\,\delta\psi_\rho(x''); \tag{30}$$

$$\psi_\mu(x)\,\delta\bar{\psi}_\nu(x')\,\psi_\rho(x'') + \psi_\rho(x'')\,\delta\bar{\psi}_\nu(x')\,\psi_\mu(x) = 0, \tag{30'}$$

are also valid. They are consistent with (22), (23), and (29).

Applying formula (30) to the relations

$$[\bar{\psi}(x)\psi(x'), G(\psi)] = i\bar{\psi}(x)\delta\psi(x') \qquad (31)$$

we obtain the following expression for the commutation relations:

$$\psi_\mu(x)\bar{\psi}_\nu(x')\psi_\rho(x'') + \psi_\rho(x'')\bar{\psi}_\nu(x')\psi_\mu(x)$$

$$= (\gamma_4)_{\mu\nu}\delta(x-x')\psi_\rho(x'') + (\gamma_4)_{\rho\nu}\delta(x''-x')\psi_\mu(x) \qquad (32)$$

etc., in agreement with (2′).

All other applications of the variational principle of Schwinger remain practically unaltered in changing the quantization method.

In concluding this section, we note that our method of quantization preserves the TCP invariance.[2] Indeed, the TCP invariance for the case of spin $\frac{1}{2}$ fields is a consequence of the antisymmetrization of the equations of motion with respect to the operator wave functions. But this antisymmetrization also lies at the basis of our method of quantization.

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Letters to the Editor

CONCERNING RYBUSHKO'S WORK, "ON THE EQUATIONS OF MOTION OF ROTATING MASSES IN THE GENERAL THEORY OF RELATIVITY"

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IN a paper by Ryabushko,[1] an attempt was made to get the equations of motion of a system of rotating bodies by Infeld's method, that is by introducing a Dirac δ function into the energy-momentum tensor. For this, Ryabushko uses the well-known formula of Newtonian mechanics by which the velocity of an arbitrary point of a rigid body a is $\mathbf{v}'_a = \mathbf{v}_a + [\boldsymbol{\omega}_a \times \mathbf{r}_a]$, where $\mathbf{v}_a$ is the velocity of the center of mass, $\boldsymbol{\omega}_a$ the angular velocity of the body, and $\mathbf{r}_a = \mathbf{r} - \mathbf{a}$ is the position vector drawn from the center of mass of the body to the point whose velocity is being determined. Ryabushko proposes to replace the expression $[\boldsymbol{\omega}_a \times \mathbf{r}_a]$ in the case when the body a is restricted to a point by the expression $\frac{1}{2}[\boldsymbol{\sigma}_a \times \nabla\delta_a]$, where $\boldsymbol{\sigma}_a$ is a pseudovector and the operator $\nabla = e_i \partial/\partial x_i$ acts on the delta function δ_a. The velocity expression thus obtained is substituted into the energy-momentum tensor [Eq. (3.4)].

However, the pseudovector $\boldsymbol{\sigma}_a$ is a combination of the spatial components of the antisymmetric angular momentum 4-tensor $S^{\alpha\beta}$. Consequently, Ryabushko supposes that the remaining components S^{0i} (i = 1, 2, 3) are identically zero. This hypothesis, according to Corinaldesi and Papapetrou,[2] can be realized for a given particle a by the introduction of the center of mass coordinates of the particle. If we have a central body of large mass whose field is given by a Schwarzschild solution, for the motion of particles of infinitesimal mass in this field we can introduce the centers of mass of these particles relative to the stationary coordinate system. However, if we have only n particles without a central body, it is impossible to introduce uniquely their centers of mass (the components of the center of mass of one particle are not components of any 4-vector). In this case, in the absence of a privileged coordinate system, one must use the full angular momentum tensor $S^{\alpha\beta}$, where $S^{0i} \neq 0$. Consequently, the energy-momentum tensor (3.4) in Ryabushko's paper cannot be correct. We shall show that even Ryabushko's equations of motion cannot be true.

In reference 1 there is obtained a correction to the equations of motion in general relativity for the case of rotating masses, which in the two-body case has the form

$$D_a^i = \frac{f m_a m_b}{c^2}\{(\dot{a}^s - \dot{b}^s)[\sigma_a \nabla(1/r_{ab}), a^s]^i - 2(\dot{a}^s - \dot{b}^s) \times [\sigma_b \nabla(1/r_{ab}), a^s]^i + (2\dot{a}^s - \dot{b}^s)[\sigma_b \nabla(1/r_{ab}), a^i]^s + (2\dot{b}^s - \dot{a}^s) \times [\sigma_a \nabla(1/r_{ab}), a^i]^s + \text{quadratic terms in } \sigma. \quad (1)$$

σ_a and σ_b are the Newtonian proper angular momenta of the bodies. The Latin indices i, k, s . . . take on the values 1, 2, 3 and refer only to the spatial coordinates. The relativistic equations of the progressive motion of the a-th body is described by the form

$$m_a \ddot{a}^i = (f m_a m_b / r_{ab})_{,a^i} + F_a^i + D_a^i, \quad (2)$$

where F_a^i is the correction to the Newtonian force in the second approximation of general relativity not counting the rotation of the bodies, as given by Einstein, Infeld, Fock, and others.

According to Fock (reference 2, p. 359) that correction to the Lagrangian in the problem of two rotating bodies which depends on the rotations of the bodies and influences their motions has the form

$$L_\omega = f c^{-2}[m_a \omega_{si}^{(b)} I_{sj}^{(b)}(3\dot{b}_i - 4\dot{a}_i) - m_b \omega_{si}^{(a)} I_{sj}^{(a)}(3\dot{a}_i - 4\dot{b}_i)](a_j - b_j)/|\mathbf{a} - \mathbf{b}|^3. \quad (3)$$

$\omega_{si}^{(a)}$ is the antisymmetric angular velocity tensor of the a-th body, and $I_{sj}^{(a)}$ is the symmetric moment of inertia tensor of the same body. The correction to the equations of motion is given by the expression

$$\partial L_\omega/\partial a_i - (d/dt)\partial L_\omega/\partial \dot{a}_i = c^{-2}[f m_a \omega_{sk}^{(b)} I_{sj}^{(b)}(3\dot{b}_k - 4\dot{a}_k) - f m_b \omega_{sk}^{(a)} I_{sj}^{(a)}(3\dot{a}_k - 4\dot{b}_k)][\delta_{ij}/|\mathbf{a} - \mathbf{b}|^3 - 3(a_i - b_i)(a_j + b_j)/|\mathbf{a} - \mathbf{b}|^5] + c^{-2}(4 f m_a \omega_{si}^{(b)} I_{sj}^{(b)} + 3 f m_b \omega_{si}^{(a)} I_{sj}^{(a)})[(\dot{a}_j - \dot{b}_j)/|\mathbf{a} - \mathbf{b}|^3 - 3(a_j - b_j)(a_l - b_l)(\dot{a}_l - \dot{b}_l)/|\mathbf{a} - \mathbf{b}|^5]. \quad (4)$$

If each body has a spherically symmetric mass distribution, we can put

$$I_{sj}^{(a)} = I^{(a)} \delta_{sj}. \quad (5)$$

Formula (1) can be written in this case as

$$D_a^i = (fm_am_b/c^2)\{[(2/m_a)I^{(a)}\omega_{si}^{(a)}(3\dot{b}^s - 2\dot{a}^s)$$
$$+ (2/m_b)I^{(b)}\omega_{si}^{(b)}(4\dot{a}^s - 3\dot{b}^s)]\,|\,\mathbf{a}-\mathbf{b}\,|^{-5}$$
$$+ (2/m_a)I^{(a)}[3\omega_{ji}^{(a)}(a_j - b_j)(a_s - b_s)(\dot{a}_s - \dot{b}_s)$$
$$+ 3\omega_{js}^{(a)}(a_j - b_j)(a_i - b_i)(2\dot{b}_s - \dot{a}_s)]\,|\,\mathbf{a}-\mathbf{b}\,|^{-5}$$
$$+ (2/m_b)I^{(b)}[-6\omega_{ji}^{(b)}(a_j - b_j)(a_s - b_s)(\dot{a}_s - \dot{b}_s)$$
$$+ 3\omega_{js}^{(b)}(2\dot{a}_s - \dot{b}_s)(a_j - b_j)(a_i - b_i)]\,|\,\mathbf{a}-\mathbf{b}\,|^{-5}\}. \qquad (6)$$

As is easily verified, formulas (4) and (6) under conditions (5) only coincide if $\sigma_a = 0$, $b_s = 0$ (a case considered by Lense and Thirring, Das, and the author, giving the motion of a satellite of small mass around a rotating central body). In the general case (4) and (6) do not coincide. Consequently, Ryabushko's assertion in reference 1 that for spherically symmetric bodies the first members of (1) coincide with the results of Fock[3] is therefore incorrect.

Formula (1) cannot be correct even under the following considerations: D_a^i from (1) and (6) can be got from Lagrange's equation

$$D_a^i = \partial L'_\omega/\partial a_i - (d/dt)\,\partial L'_\omega/\partial \dot{a}_i \qquad (7)$$

with a correction to the Lagrangian because of the rotation of the bodies

$$L'_\omega = fc^{-2}[m_a\omega_{si}^{(b)}I_{sj}^{(b)}(2\dot{b}_i - 4\dot{a}_i)$$
$$+ m_b\omega_{si}^{(a)}I_{sj}^{(a)}(2\dot{a}_i - 4\dot{b}_i)](a_j - b_j)/|\,\mathbf{a}-\mathbf{b}\,|^3 \qquad (8)$$

using condition (5). It is easily verified that, under the interchange of a and b, L'_W goes over to $-L'_W$. This fact contradicts the requirement that the correction to the Lagrangian, just like the full Lagrangian, must be invariant under the interchange of the two bodies; Fock's correction to the Lagrangian obviously satisfies this requirement.

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ON THE DEPTH OF THE POTENTIAL WELL FOR Λ PARTICLES IN HEAVY HYPERNUCLEI

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THE energy of the Λ particles in heavy hypernuclei, together with the binding energies of the light hypernuclei, imposes certain restrictions on the Λ-nucleon potential. The Λ particles in heavy hypernuclei can be regarded as moving in a square potential well whose depth is determined by the interaction of the Λ particle with the nucleons.[1,2] The author and Lyul'ka[3,4] considered Λ-nucleon potentials derived from meson theory. In order to avoid singularities at small distances, the momenta of the virtual mesons had to be cut off. These potentials yield the correct dependence of the binding energy B_Λ on the number of particles in light hypernuclei. They lead to a stronger interaction of the Λ-nucleon pair in the singlet state, which is in agreement with the value zero for the spin of ${}_\Lambda H^4$, as estimated from the ratio of the number of mesonic and non-mesonic decays. In the present note we make an estimate of the potential energy of the interaction of Λ particles in nuclear matter on the basis of the potentials obtained in references 3 and 4. In estimating the potential energy the nucleons in the nuclear matter were regarded as an incompressible degenerate Fermi gas. We carried out calculations for two values of the nuclear matter density, given by the radii $R = 1.2\,A^{1/3} \times 10^{-13}$ and $R = 1.4\,A^{1/3} \times 10^{-13}$. The actual density lies apparently somewhere between these limits.[5]

In the table we given the values of the potential energy of Λ particles in nuclear matter, U^{1K}, $U^{2\pi}$, $U^{K\pi}$, and U^{2K}, for Λ-nucleon potentials corresponding to the exchange of a single K, two π, a K and a π, and two K mesons, respectively. We also list the total potential energy U (all values are in Mev). In computing these values we assumed two types of coupling between the K mesons and the baryons: the scalar and the pseudoscalar coupling. The coupling between the baryons and the π mesons was assumed to be pseudovector with the coupling constant $f^2 = 0.08$. We used a rectangular cut-off with $k_m = 6\mu_\pi$. The resulting potential energy U

K-Baryon coupling	$R=1.4\,A^{1/3}\cdot10^{-13}$					$R=1.2\,A^{1/3}\cdot10^{-13}$				
	U^{1K}	$U^{2\pi}$	$U^{K\pi}$	U^{2K}	U	U^{1K}	$U^{2\pi}$	$U^{K\pi}$	U^{2K}	U
Pseudoscalar, $f^2=0,08$	+1	−50	+22	−20	−47	+3	−78	+34	−32	−73
Scalar, $g_\Lambda^2=g_\Sigma^2=1.1$	+17	−50	+25	−21	−29	+25	−78	+39	−33	−47
Scalar, $g_\Lambda^2=3g_\Sigma^2=1.1$	+17	−50	+17	−10	−26	+25	−78	+26	−16	−43

is negative, guaranteeing a bound state for the Λ particle. The data in the table give information on the size of the contributions from the various processes to the potential energy of the Λ particle. In view of the exchange character of the 1K and Kπ mesonic potentials, these make the interaction of the Λ particles with the nuclear matter nonlocal. However, in the approximation of "zero interaction range," we can also in this case introduce an effective potential for the Λ particles in the nucleus. The data of the table were obtained in this approximation. Owing to the effects of the Pauli principle in the system of nucleons, the contributions to the potential energy from the exchange-type 1K and Kπ forces are strongly suppressed. They are positive and have about the same absolute value as the contributions from the 2K forces.

The uncertainties in the experimental binding energies for Λ particles in heavy hypernuclei are quite large, and it is therefore impossible to obtain from them any accurate information about the well depth. Apparently, the depth should be 20 to 30 Mev for $10 < A < 20$. The depth of the square potential well in hyperfragments with $A < 10$ is about 20 Mev. The estimates of Dalitz and Downs[6] indicate that the well depth in heavy hypernuclei can be 29 to 38 Mev. The comparison of these data with those of the table shows that the scalar variant is not in disagreement with these data, while the pseudoscalar variant gives too high values for the depth, using our assumed values for the coupling constants.

Since there are at present no accurate experimental data on the binding energies of heavy hypernuclei, it is not possible to make a detailed comparison.

The author thanks Prof. D. D. Ivanenko for his support in the completion of this work.

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ON THE AXIAL ASYMMETRY OF ATOMIC NUCLEI

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To investigate the equilibrium shape of atomic nuclei, we considered previously[1] the behavior of nucleons in an infinite ellipsoidal square well with semi-axes $a_x r_0$, $a_y r_0$, and $a_z r_0$. This was done by means of a coordinate transformation which transforms the ellipsoid into a sphere of radius r_0. In the new coordinates the kinetic-energy operator of the nucleons has two parts, $-\hbar^2/2M\Delta$ and $\hat{V}$. The second part, $\hat{V}$, can be expanded into a series in powers of the deformation with a linear leading term. It is convenient to choose as the deformation parameters ρ and γ which are connected with the semi-axes by the relations

$$a_x^{-1}+a_y^{-1}-2=\rho\cos\gamma, \quad a_y^{-1}-a_x^{-1}=\sqrt{3}\rho\sin\gamma. \tag{1}$$

It is easy to see that γ coincides with the parameter γ introduced by Bohr,[2] and ρ is in first order proportional[2] to β: $\rho \approx (5/4\pi)^{1/2}\beta$. The parameters α and δ of reference 1 are connected with ρ and γ by the relations

$$\alpha=\rho\cos\gamma, \quad \delta=-\sqrt{3}\rho\sin\gamma. \tag{2}$$

Considering ρ, and consequently $\hat{V}$, to be small

the problem can be treated by perturbation methods. The degeneracy in the quantum number m is already fully lifted in the first order (see reference 1). The nucleon states now can be classified by the quantum numbers n, l, $|m|$, and w, where w is the parity with respect to reflections in the (x, z) plane. The meaning of the other quantum numbers can be easily explained by going to the limits of axial symmetry and spherical symmetry.

The energies of the levels as a function of ρ and γ was calculated up to and including quadratic terms for s, p, d, and f states. Thus, for states with $l = 3$ and $m = 1$ the energy is given by

$$E_{n31\pm} = E^0_{n3}\left\{1 + \frac{4}{15}\rho\left[\sin\left(\frac{\pi}{6} \pm \gamma\right) - \sqrt{1 + 4\cos^2\left(\frac{\pi}{6} \pm \gamma\right)}\right] + \rho^2\left[1 - \frac{2}{15}\cos^2\left(\frac{\pi}{6} \pm \gamma\right) + \frac{12\cos^3\gamma \mp \sqrt{3}\sin\gamma}{15\sqrt{1 + 4\cos^2(\pi/6 \pm \gamma)}} + \frac{24}{7}\left(5 + 4\sin^2\left(\frac{\pi}{6} \pm \gamma\right) - \frac{44\sin^3(\pi/6 \pm \gamma) - 35\sin(\pi/6 \pm \gamma)}{\sqrt{1 + 4\cos^2(\pi/6 \pm \gamma)}}\right) D^-_{n3} + \frac{72}{77}\left(103 + 60\sin^2\left(\frac{\pi}{6} \pm \gamma\right) - \frac{380\sin^3(\pi/6 \pm \gamma) - 343\sin(\pi/6 \pm \gamma)}{\sqrt{1 + 4\cos^2(\pi/6 \pm \gamma)}}\right) D^+_{n3}\right]\right\}, \quad (3)$$

where

$$E^0_{nl} = \frac{\hbar^2\mu^2_{nl}}{2Mr_0^2},$$

$$D^\pm_{nl} = \mp\left\{\frac{2(l \pm 3) + 1}{16[2(l \pm 1) + 1]^2} - \frac{\mu^2_{nl}}{8[2(l \pm 1) + 1]^3}\right\},$$

μ_{nl} is the n-th root of the spherical Bessel function $j_l(x)$. For the axially symmetric ($\gamma = 0$) case these expressions go over into Moszkowski's expressions.[3]

Numerical evaluations of the energy were performed for different configurations. They show that for the case of a few particles above the closed shell, beginning with three, the minimum of the energy can correspond to an axially nonsymmetric shape of the nucleus. So, for example, the configuration $(1s)^6$ $(1p)^6$ $(1d)^4$ of one nucleon kind, corresponding to the nucleus Mg^{24} the minimum of the energy occurs at $\beta \approx 0.3$, $\gamma \approx 7°$; for the configuration $(1s)^2$ $(1p)^6$ $(1d)^{10}$ $(2s)^2$ $(1f)^2$ corresponding to Ti^{44}, it occurs at $\beta \approx 0.2$, $\gamma \approx 5°$.

In this one has to keep in mind the rough character of the utilized model and the slow convergence of the perturbation series (this is particularly so for values of γ close to zero and for small m). Therefore the obtained results can by no means aspire to be in agreement with experiment. However, it follows from the above that the independent particle model in its simplest form contains the possibility of deviations of the nuclear equilibrium shape from axial symmetry. This result is in agreement with the results of similar calculations of Gursky[4] and Geĭlikman[5] for the oscillator potential. Analogous results have been obtained for the unified model by Davydov and Filippov.[6]

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A POSSIBLE METHOD FOR THE DETERMINATION OF THE DIRECTION OF POLARIZATION OF μ^- MESONS

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IT is well known[1,2] that the law of lepton conservation will be completely verified only when it is shown experimentally that μ^- mesons produced in the decay of π^- mesons are longitudinally polarized in the direction of their motion. Dolginov[3] suggested the use, for this purpose, of the angular distribution of circularly polarized γ rays emitted in the $2p \rightarrow 1s$ transition in mesic atoms.

In the first order effect ($2p \rightarrow 1s$ transition) the expression for the angular distribution (see, e.g., reference 4) naturally depends on the degree

of polarization P of the μ^- meson prior to the transition under study. However in reality, no such isolated transitions take place in the μ-mesic atom. Previous transitions[5] lead to a depolarization of the μ^- mesons and therefore the formula given in reference 4 cannot be used for comparison with experiment. A calculation will show that it is also incorrect to take for P the residual degree of polarization in the 2p or 3d levels. Consequently it is necessary to consider the whole effect at once, as was done for example in the depolarization calculation.[5]

The mesic atom is formed by capture, from the continuum, of a μ^- meson with orbital angular momentum l_N. Thereafter, by successive emissions of Auger electrons and γ rays (the latter transitions are important since they proceed for low l), the μ^- meson cascades down to a level l_1 from which it proceeds to the level l_0 by emission of a circularly polarized γ ray. In such a process the angular distribution of the circularly polarized γ rays is given by

$$W = 1 + [3l_1(l_1+1)]^{-1/2} \times \left\{ l_1 \left[\frac{(2l_N+3)(l_N+1)}{(2l_N+1)^2} - \sum_{l=l_1+1}^{l_N} \frac{4l^2-5}{(4l^2-1)^2} \right] - \frac{(l_1+1)(2l_1-1)l_1}{(2l_1+1)^2} \right\} \tau F_1(1 l_0 l_1) P_0 \cos\theta, \quad (1)$$

where θ is the angle between the direction of emission of the circularly polarized γ ray and the direction of motion of the μ^- meson prior to capture into the orbit l_N (i.e., the direction of the beam); P_0 is the degree of the polarization of the μ mesons in the beam; and τ and F_1 are given in references 4 and 6.* The expression (1) was derived assuming a descending cascade $l_i = l_1 + i - 1$ $(i = 1, \ldots N)$. Corrections due to other channels are negligible.[5]

Setting $l_N = 14$, $l_1 = 1$, $l_0 = 0$ in expression (1) we obtain

$$W = 1 - 0.102\tau P_0 \cos\theta. \quad (2)$$

Equation (2) gives the required angular distribution of circularly polarized γ rays in a $2p \rightarrow 1s$ transition in a μ-mesic atom.

*In reference 4 τ_1, τ_2 should be replaced everywhere by $-\tau_1$, $-\tau_2$.

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DEPENDENCE OF THE ANGULAR ANISOTROPY OF FISSION ON THE NUCLEAR STRUCTURE

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THE recently-obtained experimental data on the angular distribution of the fragments in the fission of various heavy nuclei induced by particles of 10 — 40 Mev made it possible to conclude[1-4] that a connection exists between the degree of anisotropy of the angular distribution $\sigma(0°)/\sigma(90°)$ and the parameter Z^2/A of the nucleus undergoing fission. This connection is characterized by a decrease of the degree of anisotropy with increasing value of the parameter Z^2/A.

A thermodynamical interpretation of this relation is attempted in the present work.* It is assumed that, for a sufficiently large excitation of the compound nucleus, the ratio of the cross section for fission at the angles of 0° and 90° to the direction of the incident particle varies qualitatively from nucleus to nucleus according to the known expression of statistical mechanics for the ratio of velocities of two competing processes:

$$\sigma(0°)/\sigma(90°) \sim \exp(\Delta E/T),$$

where ΔE is the difference of the activation energy of fission parallel and perpendicular to the beam, arising as a result of an interaction of incident particles with the target nucleus, and T is the temperature of the nucleus in the state of critical deformation. There are reasons for assuming that, for the nuclei considered below (far away from the nearest magic nucleus Pb^{208}), ΔE is independent of the structure of the target nucleus, but probably depends on the properties of the nucleus and particularly on its parameter Z^2/A, de-

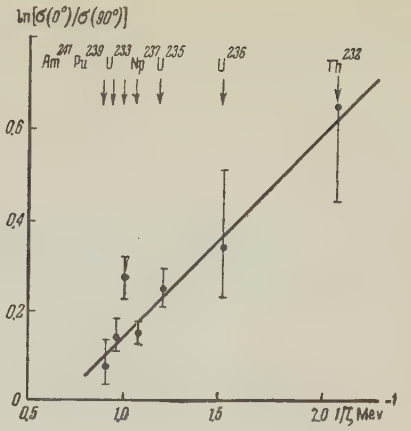
termining the cooling of the nucleus of the nucleus undergoing fission due to the evaporation of neutrons before the fission and to the loss of energy needed for producing the critical deformation. It seems probable that the last fact may be one of the causes of the dependence of the anisotropy on the structure of the nucleus.

In connection with the above, the temperature of various nuclei before fission, which is related to the excitation energy E by the expression[5] $T = 2(E/a)^{1/2}$ (where $a = 3.4(A-40)^{1/2}$ Mev^{-1}) was estimated. The effective excitation energy of a nucleus undergoing fission differs from the excitation energy of the compound nucleus (equal to the sum of the kinetic energy of the particle E_k and its binding energy E_b) by the value of the energy carried away by neutrons evaporated before fission E_{ev} and the energy lost for producing the critical deformation E_d:

$$E = E_k + E_b - E_{ev} - E_d.$$

According to the Hurwitz-Bethe hypothesis,[6] the energy E should be measured not from the ground state of the excited nucleus but from a certain characteristic level which is not influenced by the odd-even effects of protons and neutrons and the effect of filled shells. Therefore, in the determination of E, we used not the real values of E_b and E_d but the values calculated by formulae deduced from the liquid-drop model, neglecting the terms which take the effects of parity and shells into account. To determine the energy E_f, it is necessary to know the number of neutrons emitted by the excited compound nucleus before attaining the critical deformation stage. The estimate of the number of neutrons evaporated before the fission was based on the calculated relative probabilities of neutron emission and of fission.

The temperature of nuclei and the state of critical deformation were determined in the above way for the fission of Th^{232}, U^{238}, U^{235}, Np^{237}, Pu^{239} and Am^{241} induced by neutrons with energies from 14.3 to 14.8 Mev. Using the data of the references 1, 2, and 4, we plotted the dependence of $\ln[\sigma(0°)/\sigma(90°)]$ on 1/T, as shown in the figure. The fact that the experimental data on the anisotropy fall on a straight line (within the limits of errors) indicates that $\sigma(0°)/\sigma(90°)$ really varies from nucleus to nucleus as $\exp(\Delta E/T)$, for ΔE = constant, and that $T = f(Z^2/A)$. It seems therefore probable that the observed decrease of anisotropy with increasing parameter Z^2/A is, to a certain degree, connected with the fact that the temperature of a nucleus before fission increases with increasing parameter Z^2/A and, consequently, that the particles incident in the parallel direction are less important as a factor inducing fission.



*After concluding this work, the author became aware of the paper of Halpern and Strutinskii, presented at the Second International Conference on the Peaceful Use of Atomic Energy (Geneva, September 1958, Paper P/1513). On the basis of the theory on the angular anisotropy of the fission presented there, the authors gave an explanation of the dependence of the anisotropy of the structure of the nucleus which, in certain respects, is similar to the one given here.

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DIRECT INTERACTION IN REACTIONS WITH EMISSION OF TWO NUCLEONS

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In collisions of a nucleon having an energy larger than 10 Mev with a nucleus, a direct interaction of the incident nucleon with one of the nucleons of the nucleus is possible, which leads to the emission of both nucleons. This process was found experimentally, e.g., in reactions (n, 2n), (p, p'n)[1,2] and others. The angular distribution for one of the produced nucleons and the total cross section for such processes were calculated by Austern et al.[3] and Mamasakhlisov,[4] who assumed the wave functions of nucleons produced by the reactions to be plane.

We consider here the momentum angular distribution of the c.m.s. of the two nucleons produced in a direct interaction of an incident nucleon with a nucleon of the nucleus in reactions of the type (n, 2n), (p, 2p), (n, n'p), (p, p'n). The calculation was carried out under the following assumptions: The wave function of the incident nucleon (n_0) is assumed to be plane. The wave function of the nucleon (n_1) inside the nucleus with which the incident nucleon interacts is based on the shell model, assuming LS coupling. The interaction between nucleons $V_{n_0 n_1}$ is taken in the form of a rectangular well with radius ρ_0, while ψ_{2n}, the wave function of the system of two nucleons which takes into account the interaction of nucleons in the final state for $\rho > \rho_0$, is assumed to be of the form[5]

$$\psi_{2n}^{(1)} = \exp(i\mathbf{f}\rho) + a\rho^{-1}\exp(-if\rho),$$

where $\mathbf{f}$ is the wave vector of the relative motion of the nucleon n_0 and the nucleon n_1, and $a = -(\alpha - if)$ is the scattering length [$\alpha = (M\epsilon\hbar^{-2})^{1/2}$, where ϵ is the interaction energy of the two nucleons]. Inside the interaction region ($\rho < \rho_0$) the radial part of the wave function of the system of two nucleons is of the form $\psi_{2n}^{(2)} = A\rho^{-1}\sin k'\rho$, where $\mathbf{k}'$ is the wave vector with respect to the motion of n_0 and n_1 inside the potential well. A and k' are determined from the junction of the functions $\psi_{2n}^{(1)}$ and $\psi_{2n}^{(2)}$ at the point $\rho = \rho_0$. The interaction of emitted nucleons with the residual nucleus is neglected.

Under the above assumptions, the differential cross section for the process under consideration can be written in the form

$$\frac{d\sigma}{d\Omega} = \frac{M_0 M_{2n}}{(2\pi\hbar^2)^2}\frac{k_{2n}}{k_{n_0}}\frac{1}{(2s_0+1)(2J_1+1)}|I|^2, \qquad (1)$$

where M_0 and M_{2n} are the reduced masses of the incident nucleon and the system of two nucleons, k_{n_0} is the momentum of the incident nucleon, k_{2n} is the momentum of the c.m.s. of the two nucleons, s_0 is the spin of the incident nucleon, J_1 is the total moment of the original nucleus, $|I|^2$ is the square of the matrix element summed over all final states, and

$$I = \sqrt{n}\langle l^n, \alpha_1 L_1 S_1 J_1 T_1; \; k_{n_0 s_0}|V_{n_0 n_1}|\, l^{n-1}, \; \alpha_2 L_2 S_2 J_2 T_2; \; k_{2n} f_{2n} s_{2n}\rangle, \qquad (2)$$

where index 1 refers to the original nucleus, index 2 to the final nucleus, s_{2n} is the spin of the system of the two nucleons, and T is the isotopic spin.

The wave function of nucleon n_1 is separated from the wave function of the initial nucleus by means of the parentage and the Clebsch-Gordan coefficients. The summation over the magnetic numbers of the final state is carried out by a method similar to that of reference 6. For the case $E_{n_0} \leq 20$ Mev, the basic contribution to the differential cross section will be due only to the s wave in the partial-wave expansion of the wave function of the relative motion of the two emitted nucleons. From the Pauli principle, the spin function should be antisymmetric and $s_{2n} = 0$ for two identical nucleons produced, e.g., as a result of the reactions (n, 2n) and (p, 2p). For an event for which two different nucleons are emitted, s_{2n} may be 0 and 1; consequently, in constructing the angular distribution, one has to take into account two possible spin states. After integrating the matrix element over the variables of the final nucleus r_{n_1} and ρ, the square of the matrix element will be a function of k_{2n}. For a given angle between k_{2n} and k_{n0}, the value of k_{2n} is related to the value of the momentum of the recoil nucleus by the conservation of momentum law. Consequently, the square of the matrix element must be integrated over the whole range of variation of k_{2n} with the kinetic energy of the residual nucleus.

The curves of the angular distribution of the vector k_{2n} for the reaction $Be^9(n, 2n)Be^8$ were calculated numerically. The following data were used in the calculation: $E_{n_0} = 14$ Mev, the excitation energy of the Be^8 nucleus equal to 2.9 Mev, and $\epsilon = 70$ kev. The integration was carried out over a range of variation of the kinetic energy of

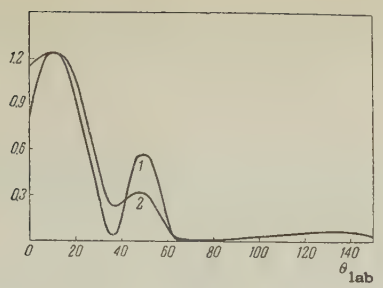

FIG. 1

Be^8 from 0.5 to 2 Mev. The calculated curves of the angular distribution k_{2n} for $\rho = 1 \times 10^{-13}$ cm (curve 1) and $\rho = 2.8 \times 10^{-13}$ cm (curve 2) are shown in Fig. 1. The histogram of the angular distribution of the vector k_{2n} obtained experimentally by using nuclear emulsions,[7] where the reaction was identified by the particle tracks produced as a result of the decay of the Ne^8 nucleon, is shown in Fig. 2. The peak for small angles was not found in the experiment. This may possibly be explained by the fact that we probably omitted those stars, in which k_{2n} is directed forward with respect to the direction of the incident neutrons and the kinetic energy of the Be^8 nucleus is smaller than 1 Mev, i.e., when the tracks of α particles produced in the decay of Be^8 are difficult to observe in the emulsion. If one constructs the theoretical curve of the angular distribution of the vector k_{2n} for the variation of the kinetical energy of Be^8 in the range 1 to 2 Mev, then the peak for small angles disappears (see Fig. 2, solid curve) and an approximate agreement between the experimental histogram and the calculated curve can be seen (curve for $\rho_0 = 2.8 \times 10^{-13}$ cm).

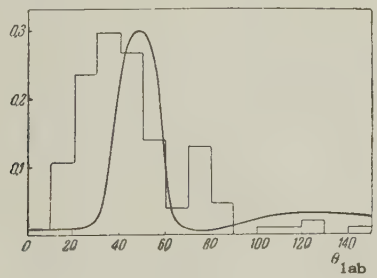

FIG. 2

Numerical calculations of the angular distribution of an event in which a bound bi-neutron is emitted yield the usual Butler-type curve with a sharp maximum at 0°, and with first minimum at 30°. Evidently, this curve cannot be compared with the histogram since, as has been shown above, it is difficult to observe in the emulsion events with emission of two neutrons in the forward direction.

In conclusion, the authors would like to express their gratitude to S. S. Vasil'ev for discussing the results, and to V. G. Neudachin for helpful advice and remarks.

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ON THE INFLUENCE OF THE ISOBARIC STATE OF A NUCLEON ON THE ELECTRON-NEUTRON INTERACTION

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IT is well known that the experimentally-found large value of the electromagnetic radius of the proton on one hand, and the very small depth of the well of the electrostatic electron-neutron interaction on the other, lead to considerable difficulties in interpretation of these data.

Since it would have been very undesirable to abandon the idea of charge independence of strong interactions, the assumption has been proposed that the usual electrodynamics ceases to be correct at distances of the order of 10^{-14} cm. It could then be understood why large electromagnetic dimensions are observed for a proton for a small depth of the well $V_0^{(S)}$ of the electrostatic e-n interaction without abandoning the charge independence of meson-nucleon interactions.

However, the theoretical explanation of the observed value of $V_0^{(S)}$ is, in itself, rather difficult. It is known that the relativistic pseudo-scalar theory in the lower order cannot give a reasonable value of $V_0^{(S)}$ from the constant of the N-π interaction. This result is obviously not unexpected since perturbation-theory calculations are simply not applicable in this region. In connection with the above, one has to bear in mind that in this order of perturbation theory one obtains a wrong value of the ratio of the magnetic moments of the neutron and the proton. It is true that the relativistic theory with cut-off[3] enables us to attain a certain improvement in the value of the ratio $\mu^{(n)}/\mu^{(p)}$, but in that case, $V_0^{(S)}(N\pi) = (f^2/4\pi)\,46.3$ kev, and for $f^2/4\pi = 0.08$ the depth of the well is found to be equal to 3.7 kev, which is even worse than the result obtained in the relativistic theory without cut-off.

The success of the semi-phenomenological isobaric theory[4-6] is well known. This theory describes sufficiently accurately a rather wide range of effects connected with meson-nucleon interactions. This can be explained by the fact that the isobaric theory is based on the certainly correct assumption of a very strong interaction of the meson-nucleon system in the state $T = J = 3/2$. It should be noted that with the isobaric theory one can obtain satisfactory values of the anomalous magnetic moments of the nucleons.[7] An attempt has been made[8,9] to take the influence of multimeson states on the electron-neutron interaction into account, and it was found that the effect is qualitatively favorable. The authors made an essentially similar attempt of a fuller explanation of the characteristic features of meson-nucleon interactions, and in particular of the resonance character of the interaction in the state with $T = J = 3/2$.

In view of the above, one can expect that, in calculations using a Hamiltonian containing a term corresponding to the excitation of the nucleon isobar with $T = J = 3/2$, one can reduce the present discrepancy between the observed and theoretical values of $V_0^{(S)}$.

To estimate the influence of isobaric states on the electro-magnetic radius of the nucleon, the triangular diagrams without fermionic pairs were considered, and the motion of the isobar was neglected in the operator of the spin projection. It was assumed that the nucleon and the isobar have opposite internal parities, i.e., the Hamiltonian describing the excitation of the isobaric state is given by the formula

$$H'_{I\pi} = (F/\mu)\,[\bar{\Psi}_\nu T_\alpha \psi \partial\varphi_\alpha/\partial x_\nu + \bar{\psi} T_\alpha^{+} \Psi_\nu \partial\varphi_\alpha/\partial x_\nu], \qquad (1)$$

where Ψ_ν is the spin vector corresponding to the particle with spin $3/2$, and the four-row matrices T_α take care of the isotopic invariance of the interaction.

As a result of the calculations of the above diagrams, one obtains the following expression for the mean-square electric radius of the nucleon, resulting from the presence of isobaric states:

$$\langle r^2\rangle_{I\pi}^{(N)} = (F^2/4\pi M_N^2)\,[0{,}8 - 12.2\tau_z], \qquad (2)$$

where the momenta of virtual particles are cut off at a value of the order of the nucleonic mass. The depth of the electrostatic e-n well is

$$V_0^{(s)}(I\pi) = -1.42 M_N^2 \langle r^2\rangle_{I\pi}^{(n)} \text{ kev}.$$

Comparing this with the result obtained in the usual pseudo-scalar theory with cut-off and putting $F^2/4\pi = 0.15$, we have for the resulting well depth

$$V_0^{(s)} \approx 1 \text{ kev},$$

which is much better than the former value. For the same selection of free parameters, one obviously cannot expect very good values of magnetic moments. Keeping in mind the approximate nature of the calculation, the numerical results given are only to be considered as qualitative.

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COMMENTS ON THE COVARIANTS IN THE BETA-DECAY INTERACTION

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IN the description of weak interactions, in particular the β-decay interaction, for which the interaction Hamiltonian density is

$$H' = \sum_i H_i' = \sum_i g_i (\bar{\psi}_p O_i \psi_n)(\bar{\psi}_e O_i (1 + \lambda_i \gamma_5)\psi_\nu) + \text{Herm. conj.}$$
$$i = S, V, T, A, P, \qquad (1)$$

only first order perturbation theory is normally used. This is justified by the weakness of the interaction, i.e., by the smallness of the dimensionless expansion parameter $\beta = gK^2/(2\pi)^2$ if a momentum cut-off K is introduced ($\hbar = c = 1$). This condition is satisfied if $r_0 = 1/K \sim 1/m_\pi$ or $\sim 1/m_N$ (m_π and m_N are the π meson and nucleon masses respectively) for $g \sim 10^{-32}$ cm^2. However it is conceivable that shorter distances are relevant[1] for weak interactions, i.e., higher order corrections may have to be taken into account.* (This question was first discussed for the β-decay interaction by Heisenberg.[2])

Polubarinov[3] investigated n-p scattering due to an interaction of the form $(\bar{p}p)(\bar{n}n)$ and found that the inclusion of higher order approximations substantially modified the results of the first order approximation.

In this note we consider the effect of higher order approximations to interaction (1) on the β decay of the neutron and show that for certain interaction covariants the form of the matrix element does not change from that given by first order perturbation theory. This statement is also valid for all other processes which are allowed by the first order approximation to H', e.g., $\tilde{\nu} + p \to n + e^+$, provided that one also takes into account the interaction $(\bar{p}n)(\bar{\mu}\nu)$ for the indicated processes, taken in the same form as (1).

Let us assume that $K \gg m_N$, since for $K \sim m_N$ the contribution of higher order terms is small for a conventional value of g. This condition, together with the assumption that the energy of the particles in the center of mass system is $\ll K$, permits one to ignore logarithmically divergent terms in the integrals compared to quadratically divergent ones. For example, in a typical integral

$$\int \frac{\mathrm{Sp}\{[i\gamma(q + p/2) - m_1] O_i [i\gamma(-q + p/2) - m_2] O_j\}}{[(q + p/2)^2 + m_1^2][(q - p/2)^2 + m_2^2]} d^4q$$

we can ignore the term proportional to $m_1 m_2$. Hence the S-matrix will not contain any terms with a linear dependence on the mass of any particle. This fact makes it possible to show that the S matrix for the β decay of the neutron, in the case of interest to us when only a single covariant j is involved ($H' = H_j$), will have the form

$$S^{(j)} = g_j \sum_i \varphi_i^{(j)} (\bar{\psi}_p O_i \psi_n)(\bar{\psi}_e O_i (1 + \Lambda_i^{(j)} \gamma_5)\psi_\nu), \qquad (2)$$

where $\varphi_i^{(j)}$, $\Lambda_i^{(j)}$ are scalar functions of $|g_j|^2$, λ_j, K and the invariants formed from the 4-momenta of the particles (they depend on $|g_j|^2$ because for the processes under study only odd approximations to H' are relevant); ψ_n, ψ_ν, etc., are the field operators in the interaction representation.

Let us introduce the transformation

$$\psi_p \to \gamma_5 \psi_p, \quad \psi_n \to \gamma_5 \psi_n, \quad m_p \to -m_p, \quad m_n \to -m_n,$$
$$g_j \to \varepsilon_j g_j, \quad \lambda_j \to \lambda_j, \qquad (3)$$
$$\bar{\psi}_p \to -\psi_n^T C^{-1}, \quad \psi_n \to C\bar{\psi}_p^T, \quad m_p \rightleftarrows m_n,$$
$$g_j \to -k_j g_j, \quad \lambda_j \to \lambda_j. \qquad (4)$$

$m_n = m_p$; ϵ_j and k_j are determined from the relations $\gamma_5 O_j \gamma_5 = -\epsilon_j O_j$ and $C^{-1} O_j C = -k_j O_j^T$. The superscript T denotes transposition of spinor indices. The transformations (3) and (4) leave the interaction Hamiltonian and the commutation relations invariant and should not change the form of $S^{(j)}$. (The free field Lagrangian may be written in a form invariant under (3) and (4).) It therefore follows that for j = V, T, A in the sum (2) only the one term with $\varphi_j^{(j)}$ survives. For j = S both $\varphi_S^{(S)}$ and $\varphi_P^{(S)}$ may be different from zero. An estimate of the "admixture" $\varphi_P^{(S)}$ using perturbation theory gives zero in third order and $\varphi_P^{(S)} < 0.01\,\varphi_S^{(S)}$ in fifth order. The situation is analogous for j = P.

If $\lambda_j = \pm 1$ then it follows from the invariance of the S matrix under the transformation $\psi_\nu \to \pm\gamma_5\psi_\nu$ that $\Lambda_i = \pm 1$ respectively.

In the case of the V-A interaction[4] it is easy to show, using relation (2) and the symmetry of the Hamiltonian under permutations, that again no difference from first order perturbation theory results. The same is true for the combinations S + P − T, 3(S+P) + T.

Consequently, even if higher order corrections to the β decay interaction are important, under our assumptions it is still possible to decide from experiment what covariants are involved in the Hamiltonian (1).

I am grateful to Prof. M. A. Markov for suggesting this problem and for his interest in the work, and to I. V. Polubarinov, M. I. Shirokov and Chou Kuang-Chao for discussions.

*This possibility, although it is not without difficulties, is of interest because it may reduce the number of Hamiltonians of type (1).

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ALPHA DECAY CONSTANTS OF NON-SPHERICAL NUCLEI

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IN references 1—4, the author developed an analytical theory of alpha decay of non-spherical nuclei,* and simple formulas were given for the relative intensities of the fine structure lines. References 1 and 2 give expressions for wave functions which also make it possible to calculate the absolute decay probability. For even-even nuclei we obtain:

$$W = \frac{3\hbar\varkappa}{mR_0} w_\alpha \Gamma \left| \int_0^1 e^{\beta P_s(\mu)} d\mu \right|^2 \sum_J w_J, \qquad (1)$$

Where W is the decay probability per unit time; $\sum_J w_J$ is the sum of the relative decay probabilities to all rotation levels of the daughter nucleus, including decay to the ground state $w_0 = 1$;

$$\Gamma = \exp\left\{-2\int_{R_0}^{a_0} \sqrt{\varkappa_b^2 R_0/r - k^2}\, dr\right\}$$

$$= \exp\left\{-2\left(\frac{\varkappa_b^2 R_0}{k}\tan^{-1}\frac{\varkappa}{k} - \varkappa R_0\right)\right\};$$

$k = \sqrt{2mE/\hbar}$ is the wave number of an α particle at infinity; $\varkappa_b = \sqrt{4mZe^2/\hbar^2 R_0}$ is the wave number corresponding to the height of the Coulomb barrier for an α particle $2Ze^2/R_0$; m is the reduced mass; $\varkappa = \sqrt{\varkappa_b^2 - k^2}$; a_0 is the return point corresponding to the decay energy E to the ground state; the quantity β is given by the relation

$$\beta = [{}^4/_5 \varkappa R_0 (1 - k^2/2\varkappa_b^2) - i2k^3 R_0/5\varkappa_b^2]\,\alpha_2$$

and is connected with the quadrupole deformation α_2 contained in the equation $R(\mu) = R_0\{1 + \Sigma \alpha_n P_n(\mu)\}$ for the form of the daughter-nucleus surface.

Let us explain the meaning of the "internal probability of formation of an α-particle" w_α. The wave function of the mother nucleus can be expressed in the form:

$$\Psi = \sum_{ik} \psi_{ik}(\mathbf{r})\, \varphi_i^\alpha \varphi_k, \qquad (2)$$

where φ_i^α and φ_k are the wave functions of the stationary state of internal motion of an α particle and a daughter nucleus respectively; $\mathbf{r}$ is the radius vector of an alpha particle relative to the center of mass of the daughter nucleus. Only the term $\psi_{00}(\mathbf{r})\,\varphi_0^\alpha \varphi_0$ appears in alpha decay to the ground state. The function $\psi_{00}(\mathbf{r})$ goes over continuously to the external wave function found in reference 1. Considering the nuclear substance to be homogenous and the mean free path of an alpha particle in it to be small compared to the size of the nucleus, it is natural to assume $\psi_{00}(\mathbf{r}) = \text{const}$. Then

$$w_\alpha = \int |\psi_{00}|^2\, dV = \tfrac{4}{3}\pi R_0^3 |\psi_{00}|^2 < 1.$$

Since there must be many excited states with short alpha-particle mean free paths in sum (2), we have

$$w_\alpha \ll 1. \qquad (3)$$

With the help of formula (1), we can, from experimental data, determine w_α, for which the calculated results are shown in Table I.† Condition (3) is fulfilled only with $R_0 = 1.4\,A^{1/3} \times 10^{-13}$ cm but when $R_0 = 1.0\,A^{1/3} \times 10^{-13}$ the value of w_α increases by four or five orders of magnitude. With this, the good constancy of w_α in the entire region of alpha active nuclei confirms the reasonableness of the basic premises of the theory. Therefore, for alpha decay $R_0 \geq 1.4\,A^{1/3} \times 10^{-13}$ cm.

For non-even nuclei the wave functions derived in reference 2 give

$$W = \frac{3\hbar\varkappa}{mR_0} w_\alpha \Gamma \sum_{l=0}^{2K}{}' (2l+1) \left| C^{KK}_{KKl0} \right|^2 \times e^{-\gamma_l l(l+1)} \left| \int_0^1 e^{\beta P_2(\mu)} P_l(\mu) d\mu \right|^2 \sum_J w_J, \qquad (4)$$

where K is the projection of the nuclear moment on the axis of symmetry; the prime for the sigma means that the summation is done only for even l; $C^{JM}_{j_1 m_2 j_2 m_2}$ are the Clebsch-Gordan coefficients; $\gamma_l = 2\kappa/\kappa_b^2 R_0$. Results of a comparison with an experiment are listed in Table II. The average value of w_α is twice as small as for even nuclei.

TABLE I

Nucleus	w_α
Ra^{222}	0.155
Ra^{224}	0.051
Ra^{226}	0.068
Ra^{228}	0.104
Th^{226}	0.057
Th^{228}	0.063
Th^{230}	0.094
Th^{232}	0.079
Th^{234}	0.201
U^{230}	0.113
U^{232}	0.094
U^{234}	0.088
U^{236}	0.134
U^{238}	0.110
Pu^{238}	0.090
Pu^{240}	0.096
Cm^{242}	0.094
Cm^{246}	0.107
Cm^{248}	0.189
Cf^{250}	0.095
	$\bar{w}_\alpha$=0.10

TABLE II

Nucleus	K	w_α
Ac^{227}	$^3/_2$	0.0308
Th^{229}	$^5/_2$	0.0459
U^{235}	$^1/_2$	0.0413
Np^{237}	$^5/_2$	0.0517
Np^{239}	$^5/_2$	0.0724
Pu^{239}	$^5/_2$	0.0387
Cm^{245}	$^9/_2$	0.0361
Bk^{249}	$^7/_2$	0.0560
		$\bar{w}_\alpha$=0.047

*In a recent article[5] it is stated that the 2^4-pole interaction of an alpha particle with a daughter nucleus was not taken into consideration in any of the earlier published articles; it is also asserted that the theoretical calculations for non-even nuclei were not brought to formulas comparable with experiment. Actually, reference 2 gives the derivations of simple formulas for the case of non-even nuclei and a comparison is made with experimental results. The 2^4-pole interaction of an alpha-particle with a daughter nucleus was considered in reference 3, published at an earlier date.

†The daughter nucleus is always shown in the tables.

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BREAK-UP OF CHARGED PARTICLES BY A NUCLEAR COULOMB FIELD

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A charged particle with charge Z_a and mass number a, in flying past a target nucleus (Z, A), interacts with its Coulomb field. In some cases the energy of this interaction is sufficient to dissociate the incident particle (nuclei of deuterium, beryllium, singly ionized molecules of hydrogen, and so forth). The theory of such processes was first developed by Dancoff[1] for the break-up of fast deuterons and later was carried over without special changes to the break-up of beryllium nuclei.[2] It was assumed that the disturbance (the energy of the Coulomb interaction) is time dependent, and so they used the conclusions of perturbation theory for this case.

Although quantum-mechanical methods are used in the references cited, the intermediate calculations contain a number of approximations which can hardly be justified (see, for instance, the simplification made in computing the integral I_r and the further impulse integration in reference 1). Besides, the final results are complicated and cannot always be applied to concrete evaluation. In this connection it is interesting to note an easier way of computing cross sections of the aforementioned processes — by applying a classical analysis analogous to the derivation of the Thompson formulas for the ionization of atoms by impact.[3]

Let the complex particle (Z_a, a) be capable of decaying into the parts (Z_1, a_1) and (Z_2, a_2), and let ϵ_0 be the binding energy for this decay. Then, in interacting with the Coulomb field of the target nucleus, each of these parts will receive an additional momentum equal in order of magnitude to (see reference 3)

$$p_i \approx (2ZZ_i e^2/vb)[1 + (ZZ_a e^2/\mu_a v^2)^2]^{-1/2}; \qquad (1)$$

where the index i refers to the break-up products; v is the speed of the initial particle (Z_a, a); b is its impact parameter relative to the target nucleus, and $\mu_a = aAM_p/(A+a)$ is their reduced mass. In such a case the cross section of the examined process obviously will be equal to

$$\sigma = \pi b_{max}^2, \qquad (2)$$

where b_{max} is defined from the condition that the acquired energy of relative motion E′ of the break-up products will be equal to (or greater than) the binding energy ϵ_0 of this system, that is

$$E' = \frac{1}{2\mu}\left(\frac{a_1}{a}p_2 - \frac{a_2}{a}p_1\right)^2 \geqslant \varepsilon_0, \qquad \mu = \frac{a_1 a_2}{a} M_p. \qquad (3)$$

By substituting here p_1 and p_2 from formula (1), and solving the resulting equation for b^2 and then averaging it over all emission angles, we obtain the following expression for the cross section after some simple transformations:

$$\sigma = \frac{\pi(Ze^2)^2}{E_0\varepsilon_0}\frac{Z_1^2a_2^2 + Z_2^2a_1^2}{a_1a_2}\left(1 - \frac{\varepsilon_0}{E_0}\frac{Z_a^2 a_1 a_2}{Z_1^2a_2^2 + Z_2^2a_1^2}\left(\frac{A+a}{2A}\right)^2\right). \qquad (4)$$

Here E_0 is the kinetic energy of the incident particle, which for relativistic speeds must be replaced in formula (4) by the quantity $E_0(1+E_0/2M_ac^2)(1+E_0/M_ac^2)^{-2}$. The formula obtained for the cross section is similar to the Dancoff results calculated by numerical integration for $E_0 \sim 200$ Mev and $A \sim 100$.

Since we are interested in the case $E_0 \gg \epsilon_0$ then, from relation (4) we find (expressing E_0 and ϵ_0 in Mev),

$$\sigma \approx 6.3 \cdot 10^{-26}\frac{Z^2}{\varepsilon_0 E_0}\frac{a_2^2Z_1^2 + a_1^2Z_2^2}{a_1a_2}\text{ cm}^2. \qquad (5)$$

As can be seen from the resulting formulas, the estimate of cross sections for the processes under consideration does not present any particular difficulty. Thus for instance, for deuterons ($\epsilon_0 = 2.18$) with an energy of ~ 200 Mev and for beryllium ($\epsilon_0 = 1.7$) with an energy of ~ 100 Mev, we have $1.4 \times 10^{-28} Z^2$ and $0.45 \times 10^{-28} Z^2$ cm^2 respectively. These magnitudes differ somewhat from those calculated in references 1 and 2 but the latter are also approximations and have not been checked by experiment.

In conclusion we note that the same formulas can be used to estimate the cross section of the Coulomb break-up of the molecules H_2^+, D_2^+ and others (supposing that $Z_1 \sim 1$; $Z_2 \sim 0$; $a_1 = a_2$; $\epsilon_0 \sim 2.5$ ev), for which with energies of 20 — 30 kev we obtain $\sigma \approx 10^{-18} Z^2$ cm^2. The author is grateful to Yu. V. Kursanov who pointed out the possibility of such processes taking place in ion sources with a high degree of ionization.

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A SIMPLE METHOD OF CALCULATING THE DEGREE OF IONIZATION AND THERMODYNAMIC FUNCTIONS OF A MULTIPLY IONIZED IDEAL GAS

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THE thermodynamic functions of a gas at high temperatures, when the gas atoms are multiply ionized, are calculated on the basis of ionization equilibrium. For each pair of values of the temperature and density we must solve a nonlinear system consisting of a few nonlinear algebraic equations for ion concentrations; this requires a long calculation. Such calculations have thus far been carried out only for air[1] from 20,000° to 500,000° and from 10 to 10^{-3} times normal pressure. We here suggest a simple method for obtaining a fairly accurate estimate of the degree of ionization and of the thermodynamic functions of any gas.

We write a set of Saha equations for a gas consisting of atoms of a single element:

$$N_eN_{n+1}/N_n = (2g_{n+1}/g_n)(2\pi m_e kT/h^2)^{3/2}\exp(-I_{n+1}/kT).$$

$$N = \Sigma N_n, \qquad N_e = \Sigma n N_n; \qquad n = 0, 1, 2 \ldots Z. \qquad (1)$$

Here N, N_n, N_e are the numbers of the original atoms, n-multiply ionized atoms and electrons per cm^3, and I_{n+1} is the (n+1)-th ionization potential. The statistical weight ratio g_{n+1}/g_n for the electronic states of the ions depends on the

temperature and varies irregularly with n and with a change from one element to another. However, since it remains of the order of unity we shall assume its value to be exactly 1.

We regard the ion numbers N_n and ionization potentials I_n as continuous functions of n. We construct $I(n)$ graphically by drawing a continuous curve through the discrete values of I_n obtained from tables,[2] with $I_0 = I(0) = 0$. The system of algebraic equations is transformed into a differential equation by the assumption $N(n+1) = N(n) + dN/dn$. For each pair of temperature and density values the gas contains an appreciable number of ions of two or at most three types; the distribution function $N(n)$ is thus a narrow peak. For the mean ionic charge $\bar{n} = N_e/N$ we shall take the value of n at the maximum of $N(n)$. This becomes increasingly valid as the peak of $N(n)$ becomes narrower. Under these assumptions we obtain from (1) a transcendental equation for determining the degree of ionization or mean ionic charge multiplicity $\bar{n}$ as a function of temperature and density; after taking the logarithm we obtain*

$$I(\bar{n}+1/2) = kT \ln (AT^{3/2}/N\bar{n}), \quad A = 2(2\pi m_e k/h^2)^{3/2}. \qquad (2)$$

Because of the logarithmic dependence of the right-hand side on $\bar{n}$ two or three successive approximations are sufficient to obtain a very exact value of the root $\bar{n}$ with the aid of the graph of $I(n)$.

Approximate formulas for the thermodynamic functions are obtained from the exact formulas when, in accordance with the given simplifications, we assume the distribution of N_n, or rather $N(n)$, to be a δ function around $\bar{n}$. The internal energy per original atom and the pressure are then given by

$$\varepsilon = \frac{3}{2}\left(1+\frac{N_e}{N}\right)kT + \frac{1}{N}\Sigma N_n Q_n \approx \frac{3}{2}(1+\bar{n})kT + Q(\bar{n}); \qquad (3)$$

$$p = (N+N_e)kT \approx N(1+\bar{n})kT, \qquad (4)$$

where Q_n is the energy required for the successive stripping of the first n electrons from a neutral atom and varies along a continuous curve $Q(n)$ plotted through discrete values of Q_n. Electron excitation energy, which is usually small, is here disregarded.

The entropy per original atom is given by

$$S = k\Sigma \frac{N_n}{N} \ln \left(\frac{2\pi MkT}{h^2}\right)^{3/2} \frac{g_n e^{5/2}}{N_n} + k\frac{N_e}{N} \ln \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2e^{5/2}}{N_e}$$

$$\approx k \ln \left(\frac{2\pi MkT}{h^2}\right)^{3/2} \frac{e^{5/2}}{N} + k\bar{n} \ln \frac{AT^{3/2}e^{5/2}}{N\bar{n}}. \qquad (5)$$

The electronic statistical weight $g(n)$ of the ions is set equal to unity. When S = const the resulting equation

$$\frac{T^{3/2}}{N} \exp\left\{\bar{n}\left(\frac{I(\bar{n}+1/2)}{kT} + \frac{5}{2}\right)\right\} = \text{const} \qquad (6)$$

combined with (2) represents the equation of the adiabate $T = T(N, T_0, N_0)$ in parametric form (with $\bar{n}$ as the parameter).

The same method can easily be applied to a gas consisting of a mixture of two or more elements.

The accuracy of the method is illustrated by the table, which contains the results of our computations, for air, of the average number of particles per atom $(1+\bar{n})$ and the internal energy per atom in ev (these are the upper values), compared with the results obtained by Selivanov and Shlyapintokh (the lower values) at a few temperatures and densities. For simplicity the gas was assumed to consist of a single element and the ionization potentials I_n were averages according to the nitrogen-to-oxygen ratio of the air. The molecular dissociation energy was added to the internal energy. The table shows entirely satisfactory agreement even for low degrees of ionization (when the largest errors should result).

T, °K	$1+\bar{n}$	ε	$1+\bar{n}$	ε	$1+\bar{n}$	ε
	$\rho/\rho_{norm}=1$		10^{-1}		10^{-2}	
30000	1.68	16.6	1.97	21.6	2.3	33
	1.77	23	2.04	27.6	2.21	33
50000	2.4	40.5	2.85	58.5	3.35	83
	2.42	47.8	2.85	64	3.26	80
100000	3.72	126	4.47	186	5.1	243
	3.75	140	4.45	190	5.16	252

I take this opportunity to thank Ya. B. Zel'dovich for his interest in this work and V. S. Imshennik for valuable discussions.

*The ionization potential in the Saha equation (1) bears the index $n+1$ with $I_0 = 0$. Often I_n is written, in which case I_0 is the first ionization potential; this is merely a matter of notation. When we go over to the continuous functions $N(n)$ and $I(n)$ the value of n depends on the choice of notation. A comparison with the exact calculation shows that the mean potential pertains accurately to the middle of the interval $\bar{n}$, $\bar{n}+1$ and the $I(n)$ curve is connected to the point $I(0) = 0$.

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POLARIZATION TENSORS IN THE BORN APPROXIMATION

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IT has been repeatedly remarked by various writers that the polarization of elastically-scattered nucleons, calculated in first Born approximation, is zero. It is also of interest to examine in the Born approximation the polarization of particles with higher spins.

The formal reason for the appearance of new selection rules for the polarized states is the Hermitian nature of the scattering amplitude in the Born approximation. If, in the unitarity relation for elastic scattering (cf. reference 1)

$$M(\mathbf{n}_2, \mathbf{n}_1) - M^+(\mathbf{n}_1, \mathbf{n}_2) = \frac{ik}{2\pi}\int M^+(\mathbf{n}_2, \mathbf{n})\, M(\mathbf{n}_1, \mathbf{n})\, d\mathbf{n}$$

we neglect the right member, as a quantity quadratic in the scattering amplitude, we get

$$M(\mathbf{n}_2, \mathbf{n}_1) = M^+(\mathbf{n}_1, \mathbf{n}_2). \tag{1}$$

If the scattering is described by a (Hermitian) Hamiltonian, this relation also follows from the fact that the scattering amplitude is the matrix element of the Hamiltonian calculated with plane waves.

Similarly, when besides elastic scattering there is also possible the reaction

$$a + a' \to b + b', \tag{2}$$

the unitarity relations for the scattering and reaction amplitudes given in reference 3, after neglect of terms quadratic in the amplitudes, provide the relations

$$\begin{aligned} M_{aa}(\mathbf{n}_2, \mathbf{n}_1) &= M^+_{aa}(\mathbf{n}_1, \mathbf{n}_2), \\ k_a M_{ba}(\mathbf{n}_2, \mathbf{n}_1) &= k_b M^+_{ab}(\mathbf{n}_1, \mathbf{n}_2), \\ M_{bb}(\mathbf{n}_2, \mathbf{n}_1) &= M^+_{bb}(\mathbf{n}_1, \mathbf{n}_2). \end{aligned} \tag{1a}$$

Here M_{aa} and M_{bb} are the amplitudes for the elastic scatterings $a + a' \to a + a'$ and $b + b' \to b + b'$, respectively; M_{ba} is the amplitude for the reaction (2), and M_{ab} is the amplitude for the inverse reaction.

Following references 2 and 3, we shall characterize the spin states of the particles before and after the reaction by the polarization tensors $\rho^{(0)}_{JMJ'M'}$ (before the reaction) and $\rho_{KNK'N'}$ (after the reaction). The values of JM and J'M' show the rank and projection component of the polarization tensors for particles a and a', respectively; the values of KN and K'N' show the rank and projection component of the polarization tensors for particles b and b'. As has been shown in the papers referred to,

$$\rho_{KNK'N'} = \frac{1}{\sigma_{ba}} \sum_{JMJ'M'} K^{JMJ'M'}_{KNK'N'}(\mathbf{n}_2, \mathbf{n}_1)\, \rho^{(0)}_{JMJ'M'},$$

where

$$\sigma_{ba} = \mathrm{Sp}\{M_{ba}\, \rho^{(0)} M^+_{ba}\};$$

$$K^{JMJ'M'}_{KNK'N'}(\mathbf{n}_2, \mathbf{n}_1) = \mathrm{Sp}\{M_{ba}(\mathbf{n}_2, \mathbf{n}_1)\theta_{JM}\theta_{J'M'} \times M^+_{ba}(\mathbf{n}_2, \mathbf{n}_1)\, T^+_{KN} T^+_{K'N'}\},$$

and also there follows from time reversibility the following relation for the coefficients K:

$$\begin{aligned} &k_a^2 K^{JMJ'M'}_{KNK'N'}(\mathbf{n}_2, \mathbf{n}_1) \\ &\quad = (-1)^{J+M+J'+M'+K+N+K'+N'} k_b^2 K^{K-N\ K'-N'}_{J-M\ J'-M'}(\mathbf{n}_1, \mathbf{n}_2). \end{aligned} \tag{3}$$

The operators θ_{JM}, $\theta_{J'M'}$, T_{KN}, $T_{K'N'}$ are irreducible tensor operators in the spin spaces of the respective particles a, a', b, b'.

From the condition (1a) of the Hermitian nature of the scattering amplitude and the Hermitian properties of the tensor operators ($T^+_{qK} = (-1)^K T_{q-K}$) it is not hard to get another relation for the coefficients K:

$$\begin{aligned} &k_a^2 K^{JMJ'M'}_{KNK'N'}(\mathbf{n}_2, \mathbf{n}_1) \\ &\quad = (-1)^{M+M'+N+N'} k_b^2 K^{K-N\ K'-N'}_{J-M\ J'-M'}(\mathbf{n}_1, \mathbf{n}_2). \end{aligned} \tag{4}$$

Comparing the relations (3) and (4) we see that $K^{JMJ'M'}_{KNK'N'} = 0$ if the sum $J + J' + K + K'$ is odd.

We have obtained the selection rule for the polarized states in the reaction (2). It is not hard to get from the first and third of the relations (1a) a similar selection rule for the elastic channels of the reaction.

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M1-TRANSITION IN V^{51} AND THE SENIORITY QUANTUM NUMBER

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As shown by Neudachin[1] the measurement of M1-transition probability in nuclei in which the $1f_{7/2}$ shell is being filled should allow us to answer the question of the accuracy of the seniority quantum number v for this type of nucleus. If v is a good quantum number, then, under certain conditions, the matrix element for M1 transitions in such nuclei could be equal to zero. In particular, if outside the filled shell there are only neutrons or only protons characterized by the configuration j^n, only those M1 transitions will be allowed that occur without a change in v. In V^{51} there is a filled neutron shell (N = 28) and three protons outside the shell Z = 20. The configuration of the ground state is $(f_{7/2})^3_{7/2}$, v = 1, and for the first excited state (321 kev) $(f_{7/2})^3_{5/2}$, v = 3. Thus, for the 321-kev M1 transition there is a change in the quantum number v ($\Delta v = 2$), therefore this transition must be forbidden if v is a good quantum number.

In the present work the lifetime of the 321 kev excited state of V^{51} is measured by the resonance scattering of γ quanta. Cr^{51} in the compound $CrCl_2O_2$ (chromyl chloride, an easily evaporated liquid) was the source of the γ quanta. We observed the scattered radiation from V_2O_5 and (for comparison) Cr_2O_3 scatterers with a scintillation spectrometer. The methods of measuring and handling the experimental data are analogous to those used in references 2—4. A source was prepared in which chromyl chloride evaporated completely at 110° C. This permitted us to measured the dependence of the resonance scattering effect on vapor pressure in the source up to nearly atmospheric pressure. The comparison of the measured dependence with that theoretically calculated (from the known dependence of density of chromyl chloride vapor on temperature) showed that atomic collisions in the source practically do not affect the magnitude of the effect up to temperatures of about 30—40° C. The lifetime measurements were made with a source of smaller mass at 10—18° C. The high specific activity of the source and the improved geometry allowed us to obtain a higher resonance scattering counting rate than Schopper did.[2]

The value $\tau = (3.1 \pm 0.8) \times 10^{-10}$ seconds was obtained for the lifetime of the 321-kev level of V^{51}. Taking into consideration data on the Coulomb excitation of V^{51},[5] this corresponds to a partial lifetime for M1 components of the mixed M1-E2 transition at 321 kev of 3.4×10^{-10} sec. The lifetime measurement in the present experiment exceeds by a factor of two the value obtained by Schopper using the method of resonance scattering of γ quanta, but is about the same magnitude as obtained using the delayed coincidence method[6] ($\tau = 4 \times 10^{-10}$ sec). The lifetime 3.4×10^{-10} sec corresponds to a forbiddenness of M1-transition in comparison with a single proton transition (Weisskopf formula) by 300 times. We note that the magnitude of forbiddenness (and consequently the magnitude of the transition matrix element) was the same as for l-forbidden transitions.[7] However, we must recognize this as mere coincidence since in this case l-forbiddenness cannot occur. The result we obtained has its natural explanation as a forbiddenness because of the seniority quantum number. Evidently, the 321-kev transition in V^{51} is the first known case of such forbiddenness. The forbiddenness is not rigorous, which indicates a proximity to the nuclear model used by Neudachin in calculations. The presence of the noted collective effects appearing in the considerable intensity of the E2 component in the 321-kev transition,[5] also explains that the forbiddenness because of the quantum number v does not have to be rigorous.

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ON A POSSIBLE DETERMINATION OF THE SIGN OF MUON POLARIZATION

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A determination of the polarization sign of μ mesons produced in the π-μ decay would decide the validity of the conservation law of leptons. In this connection it seems to us that the following possibility for the experimental determination of the polarization sign is of interest.

Longitudinally polarized μ mesons from the decay process are sent into a transverse magnetic field. The anomalous magnetic moment of the meson will cause its spin to rotate somewhat faster than its momentum. After a certain number of revolutions the meson will be transversely polarized. The necessary number of revolutions for the spin to turn away from the momentum by $\pi/2$ may be estimated as follows. The angle of "deviation" of the spin from the momentum is $\psi' = 2\pi n\,(\epsilon/m)\,\Delta\mu/\mu$, where $\Delta\mu/\mu$ is the relative anomalous moment of the μ meson and ϵ is its energy. To lowest order $\Delta\mu/\mu = \alpha/2\pi = 1.15 \times 10^{-3}$.[1] Hence for mesons with energy $\epsilon/m = 1.7$ the value of $n_{\pi/2} \approx 125$ is obtained.

The sign of the transverse polarization may be determined, for example, by measuring the asymmetry in Coulomb scattering of the meson.[2] The asymmetry in the scattering of polarized particles has a rather sharp maximum for velocities of the order of $\beta^2 \approx 2/3$ (for scattering through an angle $\pi/2$).[1] Consequently mesons of appropriate energy should be selected for the experiment. Since at such high energies the scattering cross section becomes very small for large angles one should use a scatterer with high Z and of substantial thickness (although not so thick as to stop the mesons altogether).

Let us estimate the intensity of the scattered mesons and the degree of asymmetry in the scattering. Assuming scattering into a solid angle ~ 1 we find (for Z = 80 and for a kinetic energy of the meson of 80 Mev) for the number of scatterings through $\pi/2$ per g/cm^2 of the substance: $\nu \approx 1.5 \times 10^{-5}$. Further, in a field $H \approx 10 \times 10^3$ oe the time spent by the mesons in the system will be $\approx 1.5 \times 10^{-6}$ sec. At $\epsilon/m \approx 1.7$ approximately half of the mesons that entered the system will traverse it (assuming no losses in the system). Assuming $\sim 10^4$ mesons incident on the system per sec and using 15 g/cm^2 for the thickness of the scatterer we obtain for the meson intensity $I \sim 1$ sec^{-1}.

The degree of asymmetry $a(\theta)$ is determined by the expression

$$I(\theta)/I(-\theta) = (1 + Pa)/(1 - Pa),$$

where P is the degree of polarization of the initial meson beam. For $\theta = \pi/2$, a_{max} has a value of $\approx \frac{1}{4}$. Apparently it is not difficult to obtain $P > 0.5$, so that the ratio of intensities is $\gtrsim 1.3$.

The proposed experiment may easily be combined with an experiment designed to measure the size of the anomalous magnetic moment μ.

I am deeply grateful to Yu. F. Orlov for a discussion of this problem which was very valuable to me.

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QUADRUPOLE MOMENT OF Er^{168}

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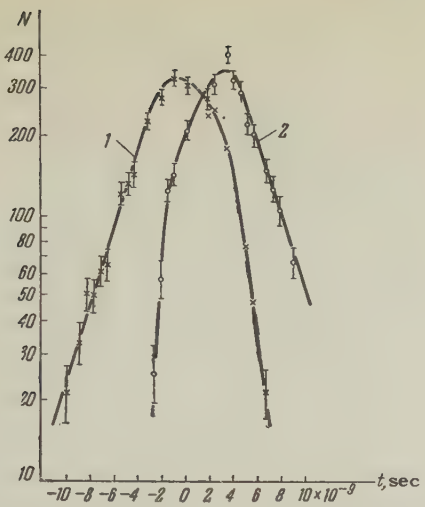
GROMOV et al.[1] and Jacob et al.[2] identified the 80-kev level of Er^{198}, produced by K capture in Tu^{168}, as the first level of the rotational band. A measurement of the lifetime of this level permits determination of the quadrupole moment and the deformation parameter of Er^{168}, using the equations of the generalized model of the nucleus.[3]

We used a weak source of Tu^{168} (T = 85 days), obtained in a deep splitting reaction by prolonged exposure of tantalum to 660-Mev protons in the synchrocyclotron of the Joint Institute for Nuclear Research.



The coincidence curves obtained in the setup described by one of the authors[4] are shown in the figure. The abscissas represent the delay time, and the ordinate the number of coincidences in logarithmic scale. Curves 1 and 2 correspond to coincidences between the x-rays that accompany the K captures and the conversion electrons produced in transitions from the 80-kev level. Curve 1 is for measurements in which the electron-absorbing filter is to the right of the source (eX coincidences), while curve 2 pertains to an absorber to the left of the source (Xe coincidences).

The errors introduced in the principal curves by x-γ, γ-x, and γ-γ coincidences, and also by multiple scattering of gamma rays in both crystals, were accounted for by measuring separately the number of coincidences as a function of the delay time, using electron absorbers placed on both sides of the source. In plotting the eX and Xe coincidences, the channel analyzers separated those parts of the spectra, corresponding to the photopeak for x-rays and to the L + M + N conversion peaks for electrons. A certain difference in shape and width of curves 1 and 2 is probably due to the different energy resolution of the two stilbene crystals. The same value of half-life is obtained from the slopes of the left and right branches of curves 1 and 2, $(1.8 \pm 0.3) \times 10^{-9}$ sec. The error given here exceeds the statistical error and takes into account possible methodical effects.

Taking into account conversion on all shells (the values of the coefficients of conversion are taken from the tables of Sliv and Band[5] for the K and L shells and from the paper of Listengarten[6] for the M + N shell) we obtain for the radiative half life

$$T_\gamma = (1+\alpha)\,T_{\text{exp}} = (15 \pm 2.5)\cdot 10^{-19}\ \text{sec},$$

where α is the total conversion coefficient.

The Weisskopf formula for a type E2 single-particle transition with energy 80 kev yields a half life of 3.10×10^{-6} sec, leading to an acceleration factor F = 200 (F is the ratio of the half-life obtained from the Weisskopf formula to the experimental radiative half-life).

The internal quadrupole moment and the deformation parameter, calculated from the Bohr formulas,[3] are equal to $(7.6 \pm 0.6) \times 10^{-24}$ and 0.32 ± 0.03, respectively. The quadrupole moment we obtained from the measurement of the half life is in good agreement with the value 7.8×10^{-24} obtained from experiments on Coulomb excitation of Er^{168} (reference 7).

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POSSIBILITY OF DETERMINING THE POTENTIAL DISTRIBUTION OF A PLASMA FROM THE CHARACTERISTICS OF THE NOISE GENERATED IN A GASEOUS DISCHARGE

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As is well known, the determination of the spatial potential distribution of a plasma by the conventional probe method involves a number of difficulties. If the discharge is inhomogeneous the probe curves exhibit discontinuities and the slope of the potential distribution curve is not easily distinguished.

The method reported here exploits noise effects characteristic of hot-cathode gaseous discharges. Frequently gaseous discharges generate noise voltages as high as 1 volt. These noise effects are observed over a frequency range ranging from kilocycles to megacycles.[1-3] It is assumed that the detection of discharge noise by a probe is affected by the space charge layer at the probe.[1,2] Since the conductivity of the plasma is higher than the conductivity of the space charge layer the impedance of the probe-cathode space is determined to a considerable degree by the impedance of the probe-plasma space which, in turn, depends only on the dimensions of the space charge layer.[4] When the probe potential is the same as the plasma potential the space charge layer around the probe vanishes. Under these conditions we may expect a sharp increase in the amplitude of the noise picked up by the probe since the conditions for detection become most favorable. Thus it is possible to determine the gas potential rapidly.

The measurements reported here were carried out in cylindrical tubes with oxide cathodes. The noise was measured in the probe-cathode section by means of a 1P-12M noise-measuring set. The experiments were carried out in krypton in the pressure region between 0.01 and 1 mm Hg at discharge currents ranging from 6 to 140 ma.

In Fig. 1 is shown a typical probe characteristic (curve marked pc) and the corresponding noise curves (the numbers on the curves are frequency in cps). The noise maxima measured at different

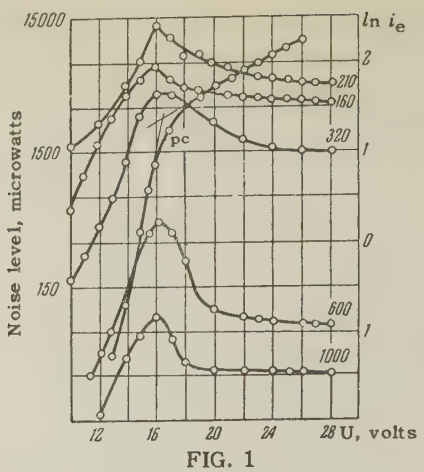

FIG. 1

frequencies are found at a probe potential of 16 volts; this value is in good agreement with the value 16.4 volts obtained from an analysis of the semi-logarithmic characteristics of the electron current to the probe (i_e). In Fig. 2 is shown the potential distribution U along the axis of the discharge as determined by the usual method (solid curve) and by the "noise" method (dashed curve). The results are in agreement to within several tenths of a volt.

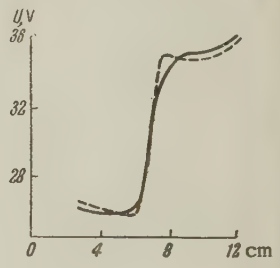

FIG. 2. The variation of potential along the axis of the discharge. The pressure is 0.1 mm Hg, the anode current is 60 ma and the cathode is to the left.

As a rule the noise curves exhibit well-defined maxima at the potentials seen by the probe. The simplicity and accuracy of the noise method make it possible to find the gas potential when the conventional probe method cannot be used to obtain an accurate determination of the discharge parameters.

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THE ANGULAR DISTRIBUTION OF NEUTRONS FROM THE REACTION $C^{13}(\alpha, n)O^{16}$

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SCHIFFER, Kraus, and Risser[1] have reported a measurement of the angular distribution of neutrons from the reaction $C^{13}(\alpha, n)O^{16}$. Measurements were made for four α-particle energies (E_α = 2.69; 2.83; 4.42 and 4.63 Mev). The final nucleus O^{16} is formed in its ground state. Furthermore, the angular distribution changes slowly as the energy of the incident α particles is varied; there are strong forward and backward peaks. All these facts suggest that the reaction is due to direct interaction. Upon colliding with a C^{13} nucleus, the α particle need not penetrate inside, but instead can "knock out" a neutron and itself be absorbed into the remaining C^{12} nucleus (in analogy with (p, n) reactions). Another process is also possible. The C^{13} nucleus can be pictured as an "asymmetric deuteron." In the collision with an α particle, the neutron can be emitted in analogy with a (d, n) reaction. Owen and Madansky[2] have used this last model in their investigation of the reaction $Be^9(\alpha, n)C^{12}$. It is clear that in the first-mentioned process, most of the neutrons will have momenta parallel to the velocity of the incoming α particles, while in the second process most of the neutrons will have momenta antiparallel.

Using Butler's results[3] to calculate the differential cross section for the "knock-on" process and those of Owen and Madansky for the second process, the following formula for the differential cross section of the reaction $C^{13}(\alpha, n)O^{16}$ can be derived:

$$\frac{d\sigma}{d\Omega} = \frac{A_1}{(K_1^2+3.05)^2}\left(\frac{1.77}{K_1}\sin K_1R_1 + \cos K_1R_1\right)^2 + \frac{A_2}{(K_2^2+0.24)^2} j_0^2(k_2R_2),$$

This formula assumes that the two processes do not interfere with each other. j_0 is the spherical Bessel function of zero order; R_1 and R_2 are characteristic radii (in units of 10^{-13} cm); and furthermore we have

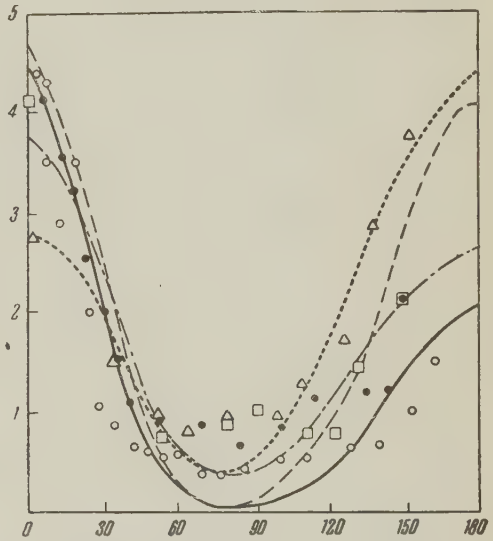
$$K_1 = \sqrt{k_\alpha^2 + k_n^2 - 2k_nk_\alpha\cos\vartheta},$$

$$k_2 = \sqrt{k_\alpha^2 + 0.06k_n^2 + 0.50k_\alpha k_n\cos\vartheta},$$

$$K_2 = \sqrt{k_n^2 + 0.01k_\alpha^2 + 0.16k_\alpha k_n\cos\vartheta}.$$

Under the square root signs, the k's are the wave numbers of the corresponding particles in units 10^{13} cm^{-1}, while θ is the scattering angle in the center of mass system. The constants A_1 and A_2 can be determined independently, since the amplitudes for the two processes do not overlap. The table shows the numerical values assigned various parameters in the calculation. The resulting, calculated angular distributions are shown in the figure, together with the experimental data.[1] (The spin of C^{13} was taken to be $\frac{1}{2}^-$, that of O^{16}, O^+ and of He^4, O^+; the odd neutron in C^{13} was taken to be in an S-state. In both processes, the angular momentum absorbed is zero.)

E_α, Mev	k_α	k_n	R_1	R_2
2.69	0.57	0.46	3.5	4.5
2.83	0.59	0.47	3.5	4.5
4.42	0.74	0.53	3.5	4.0
4.63	0.75	0.53	3.5	4.0



Neutron angular distributions. □ and the dash-dot line – E_0 = 2.69 Mev; Δ and the dashed line – E_0 = 2.83 Mev; ● and the dotted curve – E_0 = 4.42 Mev; ○ and the continuous curve, – E_0 = 4.63 Mev.

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Translated by R. Krotkov

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REMARKS ON THE OPTICAL MODEL OF THE NUCLEUS

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1. In the scattering of particles from a center of force with the potential $U(\mathbf{r})$, the scattering amplitude, $f(\vartheta)$, has the form

$$f(\vartheta) = -\frac{1}{4\pi}\int e^{-ik_0\mathbf{nr}}\, U(\mathbf{r})\,\psi(\mathbf{r})\, d\tau, \qquad (1)$$

where k_0 is the wave number far away from the scatterer, and $\mathbf{n}$ is the unit vector in the direction of the scattering. In some cases the wave function is almost undistorted near the center of force, i.e., $\psi(\mathbf{r}) \sim e^{ik_0 z}$. The substitution of this expression in (1) leads to the solution of the problem in the so-called Born approximation. In the interaction of fast particles with nuclei, $\psi(\mathbf{r})$ can often be written down in the approximation of geometrical optics: $\psi \sim e^{ikz}$, where k is the wave number inside the nucleus, which is connected with the potential U by the relation $k_0^2 - k^2 = U$. Substituting this expression in the basic formula (1), we obtain the usual optical-model solution. Both methods are therefore equivalent. However, for the investigation of any special problem, one may be more convenient than the other.

2. Let us assume that the scatterer consists of a number of identical and independent centers. If k is sufficiently close to k_0, we then obtain, after some simple transformations,

$$f(\vartheta) = f_0(\vartheta)\sum\psi(\Delta_m)\exp(-ik_0\mathbf{n}\Delta_m), \qquad (2)$$

where $f_0(\vartheta)$ is the scattering amplitude for a free single center, and Δ_m is the radius vector of the m-th center. Physically, formula (2) expresses the interference of the secondary waves from the various elementary scatterers.

We now introduce the density of elementary scatterers, $\rho(\Delta)$, and obtain

$$f(\vartheta) = f_0(\vartheta)\, F(\vartheta), \qquad (3)$$

where

$$F(\vartheta) = \int \rho(\Delta)\,\psi(\Delta)\exp(-ik_0\mathbf{n}\Delta)\, d\tau \qquad (4)$$

is interpreted as a generalized form factor. In Born approximation $\psi(\Delta) \sim e^{ik_0 z}$, and $F(\vartheta)$ goes over into the usual form factor.[1,2]

3. It can be shown that the scattering amplitude obtained by the usual optical model considerations is

$$f_{opt} = f_0(0)\int \exp(-ik_0\mathbf{n}\Delta)\rho(\Delta)\psi(\Delta)d\tau = f_0(0)\, F(\vartheta), \qquad (5)$$

this result is different from the more exact expression (3). The comparison of (3) and (5) gives

$$f(\vartheta) = f_{opt}(\vartheta)\, f_0(\vartheta)/f_0(0). \qquad (6)$$

The differential scattering cross section is

$$\sigma(\vartheta) = \sigma_{opt}(\vartheta)\,\sigma_0(\vartheta)/\sigma_0(0). \qquad (7)$$

Since $f(0) = f_{opt}(0)$, the total interaction cross section is the same in both cases, according to the optical theorem. The inelastic scattering cross section is therefore

$$\sigma_{in} = \sigma_{in\,opt} + \int \sigma_{opt}(\vartheta)\{1 - \sigma_0(\vartheta)/\sigma_0(0)\}\, d\Omega. \qquad (8)$$

The relations (6) to (8) have the same form in the laboratory system as in the center of mass system. It is evident from the physical meaning of these results, that they are also valid in the relativistic case.

4. The available experimental data on the scattering of electrons from nuclei[3] were investigated with the help of the formulas (3) and (4), in which $f_0(\vartheta)$ was replaced by the amplitude for Coulomb scattering from a point charge. The quantity $\rho_e(\Delta)$ obtained in this way gives the distribution of the charge in the nucleus. For a comparison with the data on the scattering of π mesons and nucleons from nuclei it would be more accurate to consider not the charge distribution, but the density distribution, of the nucleons, $\rho_n(\Delta)$. For this purpose one must replace $f_0(\vartheta)$ in (3) by the scattering amplitude obtained from the experiments on the scattering of electrons from protons. The corresponding correction to the quantity $\overline{r^2}$ amounts to a few percent (5.5% for C_6^{12}). The experiments on the scattering of π mesons and nucleons,[4-7] on the other hand, are usually studied with the help of formula (5). The transition to the more exact formulas (6) and (7) also involves corrections of a few percent.

For this reason, the scattering of particles of different types[4] should be compared along the lines of the above mentioned considerations.

The authors thank S. M. Bilen'kiĭ, G. V. Efimov, V. I. Ogievetskiĭ, and R. M. Ryndin for their participation in the discussion.

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ANGULAR DISTRIBUTION OF TRITONS FROM THE REACTION $Li^7(\alpha, t)Be^8$

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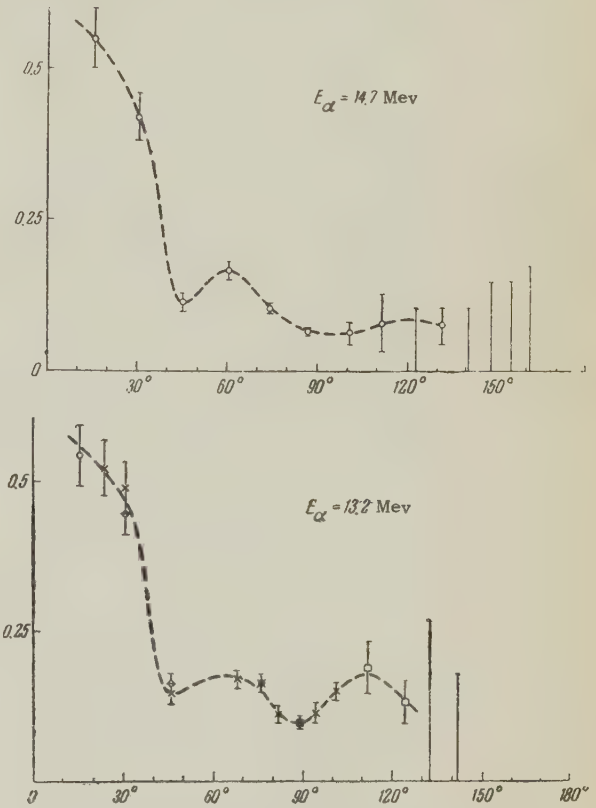
FOR the purpose of studying the reaction mechanism we used the nuclear emulsion method to investigate the angular distributions of tritons from the reaction $Li^7(\alpha, t)Be^8$ ($Q = -2.56$ Mev) with α particles accelerated in the cyclotron to energies of 8.34, 10.15, 11.5, 13.2, and 14.7 Mev. At all energies we obtained similar angular distributions. The curves in the figures show the dependence of the differential cross section (in relative units) on angle in the center of mass system with E_α = 13.2 and 14.7 Mev. The differently designated points were obtained in different experiments. At the larger angles we only evaluated the upper limit of the cross section.

The form of the angular distributions and its weak dependence on the energy of bombarding α particles show the important role of the direct interaction mechanism. Comparison with the Butler theory[1] showed that we can obtain satisfactory correspondence between theoretical and experimental curves with the angular momentum transferred to the nucleus at the time of collision $l = 1$ (the only possible value in line with the known values of nuclear spin[2] and the law of conservation of parity).

If, as is customary, we relate the isotropic part of the angular distributions, to compound nuclear processes, then we can see from the graph that the contribution of this process is large for $E_\alpha = 13.2$ Mev. We should note that in this case the full energy of motion in the c.m. system, if we include the energy spread of the α-particle beam and the energy loss in the target, corresponds to an energy ~ 16.9 Mev for the level of the compound nucleus B^{11}.



The absolute values of the differential cross section for a 16° angle (c.m.) are equal to $9.2^{+3.7}_{-1.85}$ mbn/sterad for $E_\alpha = 13.2$ Mev and $9.4^{+4.0}_{-2.0}$ for $E_\alpha = 14.7$ Mev.

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MASS OF THE ISOTOPE Pu^{239}

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A measurement of the mass of the plutonium isotope was done in a mass spectrometer[1] with a resolving power of 60,000 — 80,000. To determine the mass of the plutonium isotope we used doublets obtained with the help of organic compounds of different composition. These compounds consisted of the elements H, C^{12}, and O^{16}, whose masses had been measured earlier rather carefully.[1] By determining the isotope's mass by direct comparison with the mass of the organic compounds it was possible to avoid a series of intermediate measurements and thus significantly improve accuracy. We used two organic compounds to produce the doublet pair. In the first case we used alizarin ($C_{14}H_8O_4$, M = 240) giving fragments at mass 239. The second line of the doublet was formed by fragments of the organic compound perylene ($C_{20}H_{12}$, M = 252) having the composition $C_{19}H_{11}$. Ion formation occurred in an arc ion source whose basic discharge was maintained in helium. Pairs of plutonium and organic compounds moved into the discharge by the evaporation of these substances in crucibles of special construction.

The differences ΔM of the masses of the doublets and the corresponding value of mass of the isotope Pu^{239} are shown in the table. The final

Doublet	ΔM, mmu	Mass, Pu^{239}, mu
Pu^{239}—$C_{14}H_7O_4$	18.448±0.082	239.128922±92
$C_{19}H_{11}$—Pu^{239}	33.447±0.067	239.128695±74
		average: 239.128784±165

mass value of the isotope Pu^{239} was calculated taking into account the "weight" of the measurements. For comparison we point out that the value of the Pu^{239} mass obtained from nuclear reactions[2] (there were no mass-spectrometer measurements available up to that time) was 239.126999 ± 150.* The mass of Pu^{239} calculated from data on nuclear reactions with corrections for the more accurate value of isotope Pb^{208},[3] was 239.128025 ± 155.* The disparity between the value of the mass of the isotope Pu^{239} we obtained and the value obtained from calculations on nuclear reaction data is equal to 0.759 amu. This somewhat exceeds double the magnitude of total error of both measurements. It is interesting to note the following fact. The difference between our value and the value obtained by calculation from nuclear reaction data for the isotope U^{238} is equal to 1.035 ± 0.120 mmu,[4] and for the isotope Pu^{239} is equal to 0.759 ± 240 mmu. In addition, the deviation for the difference of the masses of Pu^{239} and U^{238} calculated by our data from the corresponding value according to nuclear reaction data is 0.166 ± 0.250 mmu, that is, it is within the limits of experimental error. In our case, the masses of the isotopes Pu^{239} and U^{238} were measured quite independently, but the nuclear measurements are connected by a continuous chain of Q values. Therefore, we may assume that the error ~1 mmu was a result of inaccurate values in the Q values that connect the reference isotope Pb^{208} with the isotopes Pu^{239} and U^{238}. This assumption is confirmed by the fact (see references 3 and 4) that deviations of difference values between our values and the nuclear values increase the farther one gets from the standard Pb^{208}, both on the side of an increase in A, and on the side of a decrease in A.

*Error actually equal to ± 1000μ mu.

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SOME CHARACTERISTICS OF THE ANNIHILATION OF AN ANTIPROTON IN THE DEUTERON

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As Pontecorvo has already noted,[1] aside from the usual annihilation of the antiproton in one of the nucleons of the deuterium nucleus, the so-called

single-meson annihilation process is also possible, where part of the energy released during annihilation is transmitted directly to the remaining nucleon

$$\bar{p} + d \to p + \pi^-; \quad (1)$$

$$\bar{p} + d \to n + \pi^0. \quad (2)$$

Obviously, the relative probability of reactions of this type will be determined not only by statistical factors, but also by the character of the annihilation interaction, of which little is yet known. Not going into the details of the existing models of the annihilation interaction, it would be natural to assume that the relative probability of these processes would be not less than the relative probability of the reaction (taking into account statistical factors)

$$\pi^+ + d \to p + p, \quad (3)$$

whose cross section for π^+ mesons with momentum ~ 130 Mev/c is $\sim 10\%$ of the total π^+d-interaction cross section.[2] At lower bombarding-particle momentum, the contribution of processes of a similar type, which include both nucleons of the deuterium nucleus, will evidently be even greater.

It is quite possible that the observable number of "no-meson" annihilation stars in emulsions ($\sim 5\%$)* is a result of the fact that in a number of cases the energy of annihilation is transmitted directly to the nucleons. For more solid conclusions we would have to have statistically richer experimental data and also an analysis of the energy distribution of nucleons emitted during annihilation.

It is easy to convince oneself that isotopic spin invariance predicts a quite definite relation between the reaction cross sections for formulas (1) and (2), namely $d\sigma_1/d\sigma_2 = 2$. We point out, however, that a deviation from this relation can be caused not only by a failure of isotopic spin invariance but also by the emission of the hypothetical ρ^0 meson with zero isotopic spin during annihilation:

$$\bar{p} + d \to n + \rho^0. \quad (4)$$

Moreover, if π^0 and ρ^0 mesons have the same structure as for instance in the Fermi-Yang model, where π^0 and ρ^0 can be described as symmetrical and antisymmetrical functions respectively of the type

$$\pi^0 = (p\bar{p} + n\bar{n})/\sqrt{2},$$

$$\rho^0 = (p\bar{p} - n\bar{n})/\sqrt{2},$$

then, in this case the relation between the reaction cross sections in (1), (2) and (4) will be $d\sigma_1 : d\sigma_2 : d\sigma_4 = 2:1:3$ (with accuracy to the relative mass differences of π^0 and ρ^0 mesons). If in this process the π^0 and ρ^0 mesons actually cannot be distinguished experimentally, then the "neutral" annihilations in formulas (2) and (4) can turn out to be twice as large as for the "charged" annihilations in (1).†

We note that besides reactions (1), (2) and (4) during the annihilation capture of an antiproton by a deutron, we can also have pair production of strange particles

$$\bar{p} + d \to \Sigma^- + K^+, \quad (5)$$

$$\bar{p} + d \to \Sigma^0 + K^0, \quad (6)$$

$$\bar{p} + d \to \Lambda^0 + K^0. \quad (7)$$

A study of the relative probability of these processes makes it possible to check isotopic spin invariance and also the correctness of the distribution of strange particles in charge multiplets associated with it. Charge independence requires that the ratio of the cross sections for reactions (5) and (6) be $d\sigma_5/d\sigma_6 = 2$. If we accept the Gell-Mann and Pais hypothesis[5] on symmetry in strong interactions, a hypothesis by which all baryons, including hyperons, emerge as isotopic doublets

$$N_1 = \begin{vmatrix} \Sigma^+ \\ Y^0 \end{vmatrix}, \quad Y^0 = (\Lambda^0 - \Sigma^0)/\sqrt{2} \text{ and } N_2 = \begin{vmatrix} Z^0 \\ \Sigma^- \end{vmatrix},$$

$$Z^0 = (\Lambda^0 + \Sigma^0)/\sqrt{2},$$

then in this case, we have the additional relation $d\sigma_6 \approx d\sigma_7$ [with an accuracy $\delta = (m_\Sigma - m_\Lambda)/m_\Lambda \approx 0.07$]. Reactions (5) — (7) can be easily identified experimentally since in this case the hyperon and the K meson are emitted at an angle of 180° with very definite energies ($E_K \approx 0.7$ Bev, $E_\Lambda \approx 0.5$ Bev, $E_\Sigma \approx 0.44$ Bev).

By similar processes, the production of cascade hyperons is also possible:

$$\bar{p} + d \to \Xi^- + K^+ + K^0,$$

$$\bar{p} + d \to \Xi^0 + K^0 + K^0.$$

I take this opportunity to express my gratitude to L. I. Lapidus, B. M. Pontecorvo, and R. M. Ryndin for their suggestions and also to D. Miller who was kind enough to give me experimental results before publication.

*The estimate of this quantity, made on the basis of available data,[3] includes corrections for $\pi^\pm$ self absorption in the emulsion nuclei, and excludes the cases when only neutron mesons are emitted (based on data on annihilation of antiprotons by hydrogen; D. Miller, private communication).

†The existing experimental data of D. Miller and others (which are still skimpy statistically) give no grounds as yet

for assuming that a noticeable number of hypothetical ρ^0 mesons is emitted, in addition to π^0 mesons, in the annihilation of the antiproton by the proton.[4]

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RESULTS OF A MODEL OF THE p-p INTERACTION AT 10 Bev

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EARLIER we developed an idea for a model of the process of multiple production.[1,2] To check the efficiency of the proposed model we made up a table of 200 random stars formed in p-p interactions at 10 Bev.[3] The table included the processes of multiple production of from 1 to 6 mesons according to statistical theory with the assumption of the existence of isobars. We are now publishing some results on the formation of stars from the table.

Figure 1 shows the obtained momentum spectrum of nucleons and mesons in the c.m. system (for comparison the smooth curve shows the same spectrum calculated in the usual way).[4] Figure 2 shows the momentum spectra of protons, π^+ and π^- mesons in the laboratory system.

The table of random stars allowed us to obtain theoretical values of the quantities for which calculation using the usual methods is difficult. Figure 3 shows the distribution of angular divergence between the charged particles in the stars in both systems (in plotting the distribution we included all $m(m-1)/2$ angles between m rays of the star). We also obtained the angular distribution between projections of rays onto a plane normal to the axis of interaction (Figure 3). This distribution is convenient because it does not change under a transformation from one system to the other. We note that with peripheral interactions, the angles $\sim 0°$ and $\sim 180°$ must occur more often than in the distribution obtained in reference 5.

Gramenitskiĭ and others[6] measured the corre-

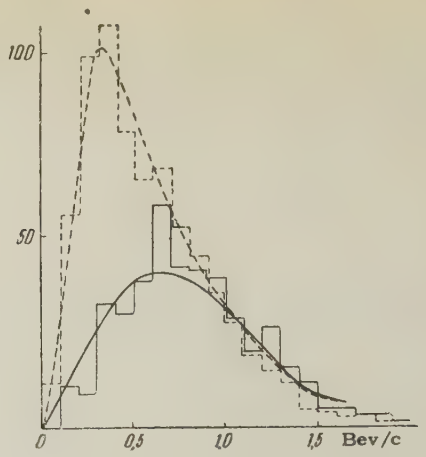

FIG. 1. Solid lines denote nucleons, broken lines denote mesons.

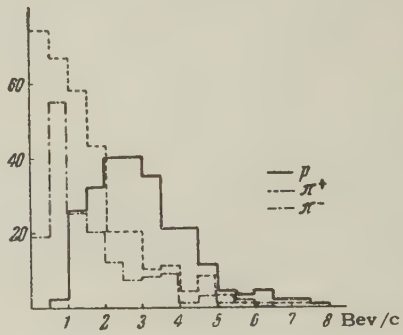

FIG. 2

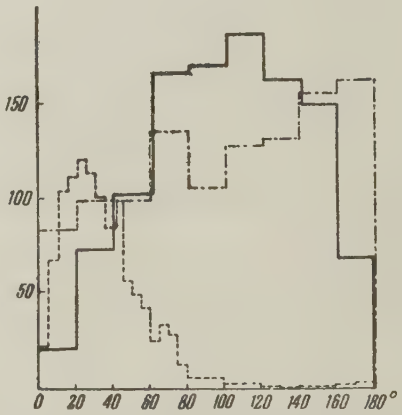

FIG. 3. Dot-and-dash lines show the spectrum of angles between projections of rays onto the plane normal to the axis of interaction; solid line shows the spectrum of angles between rays in the c.m. system; dotted line shows the same in the laboratory system.

lation coefficients between the directions of rays and the number of narrow pairs in nuclear disintegrations. Using the table of random stars it is possible to obtain the theoretical values of these quantities. The number of pairs with angle $\leq 2°$ (see Fig. 3) was equal to four in 200 stars. This is close to the quantity obtained in reference 6. The Q values (see reference 6, Table I, last line) connected with the correlation coefficients were, respectively, 0.32 ± 0.06, and 0.00 ± 0.07. A more detailed analysis of the correlations showed that as the number of rays in the stars increased, the statistical dependence between the directions of the rays becomes more obvious. This is connected with an increase of the limits placed on the directions of the particles by the law of conservation of momentum.

These results from the formation of the table of random stars testifies to the usefulness of this method for analyzing various details of the process of multiple production.

The author is grateful to M. I. Podgoretskiĭ for his interest in the work and his important advice.

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TEMPERATURE DEPENDENCE OF FERROMAGNETIC RESONANCE IN YTTRIUM FERRITE-GARNETS

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THERE have hitherto been few data on the temperature dependence of ferromagnetic resonance in ferrites. Meanwhile, the study of this dependence has acquired considerable importance, because it is here that the relation between the resonance characteristics and the magnetic properties of a ferrite is manifested most clearly. The present communication presents the results of a study of the temperature dependence of the width of the ferromagnetic resonance absorption line, the g-factor, and the resonance field in polycrystalline yttrium ferrite-garnets, in which the Fe^{3+} ions have been partly replaced by Al^{3+} and Cr^{3+} ions. At the same time, measurements of the temperature dependence of the spontaneous magnetization were made by a method described earlier.[1]

Figure 1 shows the temperature dependence of the absorption line width (solid curves) and of the specific spontaneous magnetization (dashed curves) of yttrium ferrite-garnets of the following compositions: $3Y_2O_3 \cdot 5Fe_2O_3$ (1); $3Y_2O_3 \cdot 4Fe_2O_3 \cdot Al_2O_3$ (3); $3Y_2O_3 \cdot 4.5Fe_2O_3 \cdot 0.5Cr_2O_3$ (2). Because of the small density of the specimens, the width of the resonance line was quite large; this permitted a clearer exhibition of its change on heating. It is clear from Fig. 1 that the decrease of spontaneous magnetization on approach to the Curie point occurs more abruptly than does the decrease of ΔH. It should furthermore be noted that the decrease of σ_S and of ΔH proceeds more rapidly in the case of the stoichiometric ferrite $3Y_2O_3 \cdot 5Fe_2O_3$ than it does for the "substituted" ferrites. The greater the Al^{3+} and Cr^{3+} content, the more slanting the ΔH and σ_S

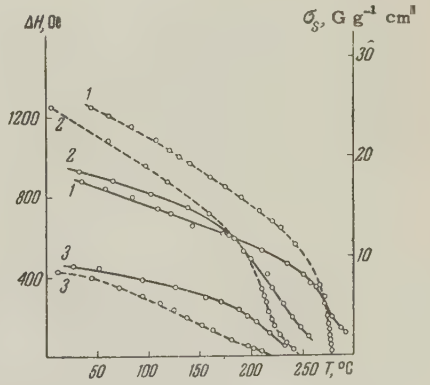

FIG. 1. Temperature dependence of the width ΔH of the ferromagnetic resonance absorption line (solid curves) and of the specific spontaneous magnetization σ, (dashed curves): 1) for $3Y_2O_3 \cdot 5Fe_2O_3$; 2) for $3Y_2O_3 \cdot 4.5Fe_2O_3 \cdot 0.5Cr_2O_3$; 3) for $3Y_2O_3 \cdot 4Fe_2O_3 \cdot Al_2O_3$.

curves are in comparison with the curves for $3Y_2O_3 \cdot 5Fe_2O_3$. Probably the Al^{3+} and Cr^{3+} ions are distributed nonuniformly in the ferrite-garnet lattice, as a result of which there occur fluctuations of the exchange interactions through the volume of the specimen; these in turn lead to a "washing out" of the curves of ΔH and σ_S vs temperature. According to the theory of Clogston, Suhl, et al.,[2] the value of ΔH in ferrites should be proportional to $\sqrt{\sigma_S}$. As can be seen from Fig. 2, this

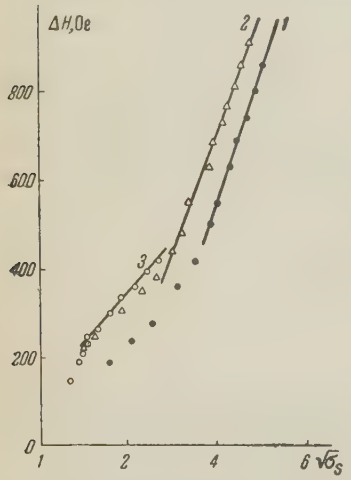

FIG. 2. Dependence of ΔH on $\sqrt{\sigma_s}$ for ferrite-garnets, calculated from the curves of Fig. 1 (same materials and same notation for them).

relation is satisfied qualitiatively in a certain temperature interval (far from the Curie point).

Figure 3 shows the temperature dependence of

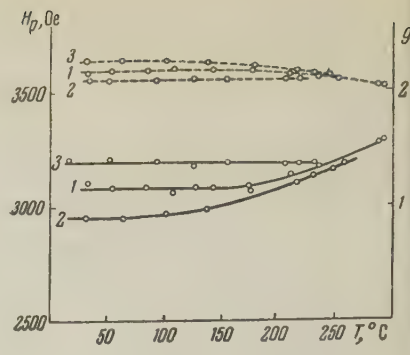

FIG. 3. Temperature dependence of resonance field H_p (solid curves) and of g-factor (dashed curves) for the same materials as in Fig. 1.

the resonance field H_p and of the g-factor on temperature. It is interesting to note that in the stoichiometric ferrite $3Y_2O_3 \cdot 5Fe_2O_3$ and in $3Y_2O_3 \cdot 4.5Fe_2O_3 \cdot 0.5Cr_2O_3$, H_p increases on approach to the Curie point, whereas the g-factor decreases slightly.

This work was performed under the direction of K. P. Belov.

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MAGNETIC AND RESONANCE PROPERTIES OF YTTRIUM FERRITE-GARNETS WITH REPLACEMENT OF Fe^{3+} *IONS BY* Cr^{3+} *AND* Al^{3+} *IONS*

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RECENTLY there has been great interest in the investigation of ferromagnetic resonance in new magnetic materials, ferrite-garnets of the rare-earth elements and of yttrium.[1,2] In the present communication we give the results of our experiments on the effect of replacement of Fe^{3+} ions by Al^{3+} and Cr^{3+} ions, in the stoichiometric yttrium ferrite-garnet $3Y_2O_3 \cdot 5Fe_2O_3$, upon the magnetic and resonance properties of this ferrite. Upon replacement of a corresponding quantity of Fe^{3+} ions by Al^{3+} and Cr^{3+}, the formulas for the ferrites studied will have the form:

$$3Y_2O_3 \cdot (5-a)\,Fe_2O_3 \cdot a\,Al_2O_3;\quad 3Y_2O_3 \cdot (5-a)\,Fe_2O_3 \cdot aCr_2O_3.$$

Here a is the content of Al^{3+} or Cr^{3+} ions per mole. The measurements of magnetic and resonance characteristics were carried out on polycrystalline specimens, prepared in accordance with standard ceramic technology (sintering at 1300° C in air for four hours). The density of the specimens was no greater than 2.75 g/cm^3; this led to a pronounced broadening of the ferromagnetic resonance absorption line. In our experiments this was an advantage, since it permitted a clear exhibition of the effect of the introduction of Al^{3+} and Cr^{3+} ions into the ferrite upon the absorption line width. For all the specimens, isotherms of the magnetization were taken by a ballistic method at helium temperatures; from these the saturation magnetization σ_0 was found, in Bohr magnetons per mole.

Replacement of Fe^{3+} ions by Al^{3+} ions (which have no magnetic moment) causes a decrease in the value of σ_0 (Fig. 1), whereas replacement by Cr^{3+} ions at first leads to an increase of σ_0; but

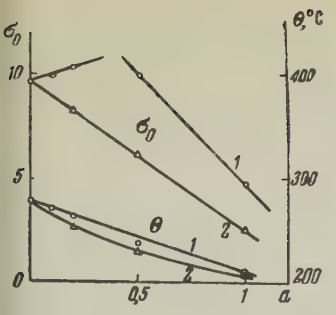

FIG. 1. Dependence of the magnetization σ_0 at 4.3°K (in Bohr magnetons) and the Curie point Θ on content a of Cr^{3+} and Al^{3+} ions in yttrium ferrite-garnet: 1) Cr^{3+}; 2) Al^{3+}.

at contents $a > 0.5$ the value of σ_0 decreases. The Curie point decreases in all cases, both on replacement by Al^{3+} and on replacement by Cr^{3+} (Fig. 1). These results are in agreement with the data of Pauthenet.[3]

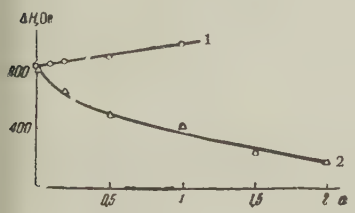

FIG. 2. Dependence of the width of the absorption line on content of Cr^{3+} and Al^{3+} ions in yttrium ferrite-garnet: 1) Cr^{3+}; 2) Al^{3+}.

Figure 2 gives the results of the measurements of ΔH, the width of the absorption line. The value of ΔH increases with increase of Cr^{3+} content, whereas it decreases with increase of Al^{3+} content. Since the magnetic anisotropy is very small in yttrium ferrite-garnet, it is possible to calculate the value of the g-factor with satisfactory accuracy. With increase of Cr^{3+} content, the g-factor rises from a value 2.150 ± 0.005 (for $3Y_2O_3 \cdot 5Fe_2O_3$) to 2.200 ± 0.005 (for $3Y_2O_3 \cdot 4Fe_2O_3 \cdot Cr_2O_3$), while upon increase of Al^{3+} content the g-factor drops to a value 2.030 ± 0.005 (for $3Y_2O_3 \cdot 4Fe_2O_3 \cdot Al_2O_3$). Comparison of the curves for ΔH, σ_0, and Θ (Figs. 1 and 2) shows that the value of ΔH is directly related to the values of σ_0 and Θ (in the case of replacement by Al^{3+}): the smaller σ_0 and Θ, the smaller ΔH. This is qualitatively in agreement with the deductions of the theory,[4] according to which ΔH in ferrites should be proportional to the values of σ_0 and of Θ. In the case of replacement by Cr^{3+} ions, as Figs. 1 and 2 show, the experimental data on the change of the values of ΔH, σ_0, and Θ are in contradiction to the theory mentioned.

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ELECTRIC FIELD IN A MICROWAVE PLASMA AS A FUNCTION OF TIME

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IT is of considerable interest to study the formation of a pulsed ultrahigh frequency (UHF) discharge from the onset of breakdown to the time at which the electron density, electric field, and other plasma parameters assume their steady-state values.

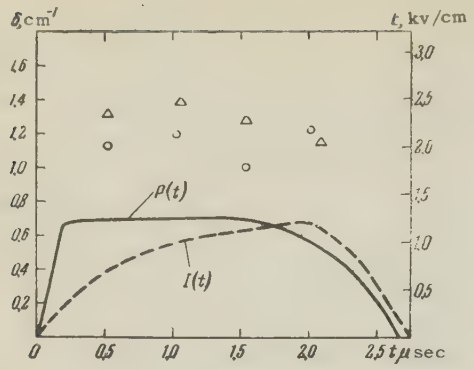
We have investigated the time dependence of the electric field during the transient period in a pulsed UHF discharge at 9400 Mcs. The field strength was measured optically by noting the Stark effect of the Balmer lines in the external oscillating field.[1,2] The microwave plasma was generated in a narrow capillary approximately 2 mm in diameter which was placed in a waveguide with cross section 23×10 mm^2.

An analysis was made of the transverse radiation with respect to the direction of the electric field. The spectral analysis apparatus was a DFS-2 diffraction grating (theoretical resolving power of 80,000). The analysis of the spectrum and the determination of the parameters of the H_β line used for the experiment were made by means of a photo-electric scanning attachment on a FEU-19 photomultiplier. To make it possible to examine the line shapes corresponding to various stages of the UHF pulses ($\tau = 2\,\mu$sec) a time-selection system was used to examine the signal from the FEU.

The triggering pulses were approximately $0.1\,\mu$sec long and could be shifted in steps of $0.1\,\mu$sec. It is assumed that the quantity being measured was essentially constant over a $0.1\,\mu$sec period. Since the repetition frequency was about 400 cps and the scanning was carried out rather slowly the results of the measurements represent averages over several thousand illumination pulses.

The measurements were carried out in deuterium at a pressure of several millimeters of mercury. In the figure is shown the time dependence of the electric field inside the plasma during a uhf pulse. In the same figure is shown the UHF pulse in the waveguide in which the capillary was located and an oscillogram of the intensity of the H_β line in relative units (same time scale for all quantities).

During the entire UHF pulse the amplitude of



P(t) – oscillogram of the power; I(t) – oscillogram of the intensity; δ – half-width of the Stark line; E – amplitude of the electric field; o – 4 mm. Hg; Δ – 23 mm. Hg.

the electric field inside the plasma remains essentially constant. On the other hand the luminous intensity, which may be interpreted as reflecting the time behavior of the electron density, shows the monotonic increase of the latter.

These results indicate that in this experiment we are dealing with the stage of discharge formation which precedes the establishment of the stationary state conditions; in the steady state the electric field is considerably smaller.[3]

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A GENERAL FORMULA FOR THE ELECTROMAGNETIC SCATTERING OF TWO DIFFERENT PARTICLES OF SPIN ½

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In view of the possibilities for experiments on the scattering of μ mesons by nucleons it is desirable to generalize the formula for the scattering of high-energy electrons by nucleons (the Rosenbluth formula) in two ways: first, to take account of the mass of the incident particle, and second, to postulate for this particle also an internal structure described by form factors $F_\mu(q^2)$ and $\Phi_\mu(q^2)$.

The charge-current density both for the nucleon and for the incident particle is taken in the form

$$\bar{u}_2 Q u_1, \text{ where } Q_i = \gamma_i F(q^2) + i\Phi(q^2)[\gamma_i, \hat{q}],$$

q is the four-vector momentum transfer; u_2 and u_1 are spinors. The square of the matrix element, averaged over the initial spin states and summed over the final spin states, is given by

$$\begin{aligned}|\mathfrak{M}|^2 = {} & \frac{1}{E_1 E_2 E_1' E_2' q^4}\Big\{\frac{1}{4}F_\mu^2 F_N^2[-q^2(M^2+\mu^2)+2(p_1\cdot p_1')^2 \\ & +2(p_1\cdot p_2')^2] + F_\mu^2\Phi_N F_N M q^2[q^2-2\mu^2] + F_\mu^2\Phi_N^2[M^2 q^4 \\ & +4q^2(p_1\cdot p_2')(p_1\cdot p_1') - 4q^2M^2\mu^2] + F_N^2\Phi_\mu F_\mu \mu q^2 \\ & \times[q^2-2M^2] + 12\Phi_N\Phi_\mu F_N F_\mu M\mu q^4 + 4\Phi_N^2\Phi_\mu F_\mu\mu q^4 \\ & \times[4M^2 - \tfrac{1}{2}q^2] + F_N^2\Phi_\mu^2 q^2[q^2\mu^2 + 4(p_1\cdot p_1')(p_1\cdot p_2') \\ & -4M^2\mu^2] + 4\Phi_N F_N\Phi_\mu^2 M q^4[4\mu^2 - \tfrac{1}{2}q^2] + 4\Phi_\mu^2\Phi_N^2 q^4 \\ & \times[4M^2\mu^2 - (M^2+\mu^2)q^2 + 2(p_1\cdot p_1')^2 \\ & +2(p_1\cdot p_2')^2 - \tfrac{1}{4}q^4]\Big\}.\end{aligned}$$

The primed quantities refer to the incident particle, the unprimed to the nucleon.

The differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{(2\pi)^2 v_{\text{rel}}}|\mathfrak{M}|^2\rho, \text{ where } \frac{e^2}{4\pi} = \frac{1}{137}, \quad v_{\text{rel}} = \frac{p_0}{E_0}.$$

In the laboratory system we have

$$E_1 = M, \quad E_2 = W, \quad E_1' = E_0, \quad E_2' = E;$$
$$(p_1\cdot p_1') = -ME_0, \quad (p_1\cdot p_2') = -ME, \quad q^2 = 2M(W-M);$$
$$\rho = p_\mu^2 W E / (p_\mu E_n - E p_0\cos\vartheta).$$

The magnitude of the momentum and the energy of the scattered meson are given by[1]

$$p_\mu = p_0\frac{(E_0M+\mu^2)\cos\vartheta + E_n\sqrt{M^2-\mu^2\sin^2\vartheta}}{E_n^2 - p_0^2\cos^2\vartheta};$$

$$E = \frac{E_n(E_0M+\mu^2) + p_0^2\cos\vartheta\sqrt{M^2-\mu^2\sin^2\vartheta}}{E_n^2 - p_0^2\cos^2\vartheta};$$

$$E_n = E_0 + M = E + W; \qquad p_0^2 = E_0^2 - \mu^2.$$

In the case in which the mass of the incident particle can be set equal to zero the expressions become much simpler:[2]

$$\begin{aligned}\frac{d\sigma}{d\Omega} = {} & \left(\frac{e^2}{8\pi E_0}\right)^2\frac{1}{\sin^4(\vartheta/2)}\left[1+\frac{2E_0}{M}\sin^2\frac{\vartheta}{2}\right]^{-1}\Big\{F_\mu^2\cos^2\frac{\vartheta}{2} \\ & \times\Big[F_N^2\Big(1+\frac{q^2}{2M^2}\tan^2\frac{\vartheta}{2}\Big) + \Phi_N F_N 4\frac{q^2}{M}\tan^2\frac{\vartheta}{2} + \Phi_N^2 4q^2 \\ & \times\Big(1+2\tan^2\frac{\vartheta}{2}\Big)\Big] + \Phi_\mu^2\Big[4F_N^2 q^2 - \Phi_N F_N\frac{8q^4}{M}\sin^2\frac{\vartheta}{2} \\ & +16\Phi_N^2 q^4\Big(\cos^2\frac{\vartheta}{2} + \frac{q^2}{4M^2}\sin^2\frac{\vartheta}{2}\Big)\Big]\Big\}.\end{aligned}$$

Setting $F_\mu = 1$, $\Phi_\mu = 0$, $\Phi_N = \kappa F_{2N}/4M$, we get the previously mentioned Rosenbluth formula.

In conclusion I express my thanks to I. L. Rozental' for a discussion.

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CONCERNING THE IMPACT OF SOLIDS AT HIGH VELOCITIES

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WHEN a solid body strikes a solid barrier at a speed exceeding several km/sec, a strong shock wave is produced in the body and in the barrier, and destruction of the crystalline lattices of the colliding bodies begins on the front of this wave. At speeds on the order of 10 km/sec and higher, (say when a meteorite strikes the surface of the moon or of an asteroid), the result is evaporation of the striking body and partial evaporation of the struck medium.

The pressure on the front of the shock wave drops sharply with increasing distance to the point of impact, and the evaporation of the medium on its front ceases and is replaced by melting and simple crushing of the medium. Strong crushing stops almost always when the mass energy density on the front of the shock becomes less than a certain value that characterizes the "strength of the medium" ϵ. The evaporated and finely fragmented medium continues to move expands, and exerts an explosion-like effect.

Since the effect of the explosion is analogous in this case to the effect of the well-studied explosion of high explosives, such as TNT, it is advantageous to introduce the so-called TNT equivalent. The equivalent mass of an explosive substance is determined by the obvious relation

$$m_s = \eta E_0 / Q = \eta M_0 u_0^2 / 2Q,$$

where E_0 is the initial energy, M_0 the mass of the striking body, u_0 the impact velocity, η the efficiency of utilization of the energy, and Q the caloric content per gram of explosive. At high impact velocity ($u_0 > \sqrt{\epsilon}$) the mass M expelled from the medium by the explosive exceeds considerably the incident mass, and therefore the momentum J of the expelled medium (normal projection) also exceeds considerably the initial momentum $J_0 = M_0 u_0 \cos\theta$, where θ is the angle (measured from the normal) at which the impact takes place.

Experiments and corresponding computations have shown that the exploded and expelled mass equals $M \approx E_0/\epsilon$. Since the momentum of the expelled mass is $J \approx \sqrt{ME_0}$, then obviously

$$J = BE_0/\sqrt{\varepsilon}.$$

Here the coefficient of proportionality B is found essentially by experiment. This coefficient can be easily related with the strength properties of the medium.[3]

The relation

$$J_0/J = 2\cos\theta\sqrt{\varepsilon}/Bu_0$$

is small at large initial impact velocities (when $u_0 > \sqrt{\epsilon}$). Consequently, the total momentum acquired by the medium during the impact and subsequent explosion depends essentially on the explosion momentum and is practically independent of the angle of the impact. At cosmic impact velocities, on the order of 30 or 40 km/sec, J exceeds J_0 by one order of magnitude.

The foregoing effects fail to apply only at very large values of the angle of incidence θ, when the normal velocity component is small.

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THE "COMPLETE EXPERIMENT" IN β DECAY

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IN an earlier paper[1] Puzikov, Ryndin, and the writer determined how many experiments are required for the reconstruction of the scattering matrix for scattering of nucleons by nucleons (the "complete experiment"). The results of that paper have been extended to the case of arbitrary spins of the colliding particles,[2] and also to the case of inelastic processes.[3] The analogous problem can also be solved for β decay.

Here we shall consider only the decay of the neutron. If the chirality of the leptons is conserved, the β-decay interaction Hamiltonian can be written in the form

$$H_\beta = (V_\alpha + A_\alpha)\langle e|\gamma_\alpha(1+\gamma_5)|\nu\rangle,$$

where V_α and A_α are the nucleon "currents" corresponding to the vector and axial-vector interactions, respectively. If we require that the neutron and the antiproton (which have the same isotopic-spin component) decay in identical ways, then it can be shown (cf. reference 4) that

$$V_\alpha = \langle p|a(k)\gamma_\alpha + b(k)\sigma_{\alpha\beta}k_\beta|n\rangle, \quad (2)$$

$$A_\alpha = \langle p|c(k)\gamma_5\gamma_\alpha + d(k)\gamma_5 k_\alpha|n\rangle; \quad (3)$$

$\sigma_{\alpha\beta} = i(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha)/2$. Here $\langle p|$ and $|n\rangle$ are the wave functions of the free nucleons, and k is the four-vector momentum transfer (the sum of the momenta of the electron and antineutrino). It can be seen from this that all the properties of the β decay of the neutron are described by the four functions a(k), b(k), ... Furthermore the coefficient b(k) describes what Gell-Mann[5] has called "weak magnetism". Ordinarily in the allowed approximation k is taken to be zero, and two constants, $a(0) = g_F$ and $c(0) = c_{GT}$ are measured. In principle, more precise measurements can also give the dependence of these coefficients on k. Since, however, the "dimensions" of the nucleon are $\sim \hbar/Mc$, and the maximum k is $\sim \hbar/mc$, the departure from the "allowed" approximation will be of the order of $m/M \sim 0.1$ percent. In such experiments we get the values of the form factors in a range of values of k determined by the decay energy. Measurements of these coefficients over a wider range of energies will be possible when it is possible to study the capture of antineutrinos of various energies by protons.

In this case four experiments must be made for each value of the momentum transfer (or for each value of the total momentum of the leptons in the β decay). The situation is simplified if we take into account a theorem of Gell-Mann,[5,6] from which it follows that for light nuclei the form factors a(k) and b(k) are the same as those that determine the scattering of electrons by protons and neutrons. Therefore if we use the data on the scattering of electrons there remain to be determined for β decay only two form factors, i.e., two experiments must be performed for each value of the momentum transfer.

It is obvious that what has been said remains valid also for any β decay between nuclei with spin $\frac{1}{2}$. If however, charge invariance is violated (non-mirror transitions), the number of form factors for each of the currents is increased by unity.

We shall also formulate the result for the general case of a transition between mirror nuclei.

If we consider the transition $I' \rightarrow I''$ (no), the number of form factors in the vector current is $2I + 1$, where I is the smaller of the spins I', I''. For light nuclei these form factors are determined from inelastic scattering of electrons with transition of the nucleus to the state isotopically similar to the final state in the β transition. The number of form factors in the axial-vector current is 2I. In the case of the transition $I \rightarrow I'$ (yes) the numbers of form factors for the two currents are 2I and $2I + 1$. These same numbers determine the number of experiments that make up a complete experiment.

In this way the study of "forbidden" approximations and forbidden β transitions determines a set of form factors. Usually forbidden transitions are described in the literature by a set of nuclear matrix elements; in reality this is only an approximate way of specifying the form factors.* Simple, but rather cumbersome calculations make it possible to express the results of the various experiments in terms of the form factors. The formulas in question will be given in a later communication devoted to the general properties of β-decay form factors.

These same considerations are also valid for the capture of mesons. At least for light nuclei it can be expected that the form factors of the vector current will be the same as for electrons (for corresponding values of k). In the case of the

axial-vector current there are indications that the β and μ form factors are different.[7] This fact makes experiments with μ mesons even more interesting.

The writer is grateful to R. Ryndin and S. Bilen'kiĭ for a helpful discussion of the results.

*We note that the form factors describe "collective" transitions which, for example, can make an important contribution in transitions of extended nuclei with large quadrupole moments.

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ANGULAR ANISOTROPY AND ENERGY CHARACTERISTICS OF THE FISSION PROCESS

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EXPERIMENTAL and theoretical investigations of the angular anisotropy of the fission of heavy nuclei, conducted so far, did not consider directly the problem of energy yield. Nevertheless, from the concepts developed up to now, one can expect that certain energy characteristics of the fission process will depend on the angle between the direction of incidence of the exciting particle and of the emitted fragments. For instance, it has been shown recently[1,2] that the degree of anisotropy in fission is strongly related to the effective temperature of the nucleus in the saddle point: the more electrons are evaporated up to the moment of attaining the critical deformation, the bigger is the anisotropy. We are, of course, considering fission of nuclei for sufficiently high excitation where the emission of nucleons before fission is energetically possible. One can then observe both cases of fission, with and without previous emission. The degree of anisotropy of the latter will be different. On the average, therefore, the number of neutrons in nuclei undergoing fission, as well as the energy of excitation, will be different for fission at an angle of 0° and 90° to the beam of incident particles. Consequently, one can expect certain differences in the kinetic energy of fragments emitted at different angles.

As an attempt to examine the relations mentioned above, the fission of U^{238} induced by neutrons with energy 14.9 Mev was studied. The energy of the additional fragments in fission along the direction of the neutron beam (0°) and in fission in the perpendicular direction to the beam (90°) was studied by means of a double ionization chamber. The angle of distribution was such that the emission direction of a fragment did not deviate by more than 26° from a given direction at 0° or 90°. Other conditions of the experiment were identical with those described earlier.[3] A total of 5000 fission events at the angle of 0° and 4000 events at the angle of 90° were recorded in alternating measurements.

It was found that, for a ratio of the fragment masses equal to 1.40 — 1.44 (close to the most probable value), the average kinetic energy of the fragments is equal to 170.8 ± 0.6 Mev at the angle of 0° and 169.4 ± 0.8 Mev at an angle of 90° (the indicated errors represent the average deviation of the results of separate series of measurements). The difference of the energy of fragments, if such exists, is therefore not bigger than 1.5%.

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ON THE RELATIVISTIC RELATION BETWEEN POLARIZATION AND ASYMMETRY

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THE usual method for determining the polarization of particles consists of measuring the azimuthal asymmetry of the scattering of the polarized particles, and is based on the fact that the azimuthal asymmetry is equal to the product of the polarization of the incident particles and the polarization produced in the scattering of unpolarized particles. This relation was derived by Wolfenstein and Ashkin for the nonrelativistic case, from general principles of symmetry.[1,2] It is interesting to examine the question of the relation between asymmetry and polarization in the case of the scattering of relativistic particles.

The state of polarization of a relativistic particle with four-vector momentum p_μ can be described[3] by the density matrix

$$\rho = {}^1/_2 (1 + i\gamma_5\gamma_\mu\xi_\mu)\Lambda_p. \tag{1}$$

Here $\Lambda_p = (\gamma_\mu p_\mu + im)/2im$ is a projection operator, m is the rest mass of the particle, and $\xi_\mu = iSp\ \gamma_5\gamma_\mu\rho$ is a spacelike axial vector orthogonal to the vector p_μ ($\xi_\mu p_\mu = 0$). The degree of polarization is defined as $P = (\xi_\mu\xi_\mu)^{1/2}$

Let us consider first elastic scattering of particles with spin $\frac{1}{2}$ by spinless particles. By considerations of invariance the scattering matrix can be written in the form

$$M = A + B\gamma_\mu(q_\mu + q'_\mu), \tag{2}$$

where q_μ and q'_μ are the momenta of the particle with spin zero before and after the scattering, and A and B are arbitrary functions of the two independent invariants. The density matrix of the final state can be put in the following form:

$$\rho_{scat} = \Lambda_{p'} M\rho_{inc}\beta M^+\beta\Lambda_{p'}. \tag{3}$$

p_μ and p'_μ are the momenta of the particles with spin $\frac{1}{2}$ before and after the scattering.

From Eqs. (1) and (3) we get the following expression for the scattering cross section of the polarized beam (polarization vector ξ_μ^{inc}):

$$\sigma = \frac{1}{2} Sp\ \Lambda_{p'} M\Lambda_p\beta M^+\beta\left(1 + \xi_\mu^{inc}\frac{Sp\Lambda_{p'} Mi\gamma_5\gamma_\mu\Lambda_p\beta M^+\beta}{Sp\ \Lambda_{p'} M\Lambda_p\beta M^+\beta}\right). \tag{4}$$

By means of the expression (2) we can verify that

$$Sp\ \Lambda_{p'} Mi\gamma_5\gamma_\mu\Lambda_p\beta M^+\beta = Sp\ i\gamma_5\gamma_\mu\Lambda_{p'} M\Lambda_p\beta M^+\beta\Lambda_{p'}$$

and, consequently,

$$\sigma = \sigma_0(1 + \xi_\mu^{inc}\xi_\mu^0), \tag{5}$$

where σ_0 is the scattering cross section of unpolarized particles and ξ_μ^0 is the polarization vector that appears from the scattering of unpolarized particles ($\xi_\mu^0 \sim \epsilon_{\mu\nu\rho\sigma}p_\nu p'_\rho q_\sigma$).

The formula (5) is also valid in the case of a reaction of the type $\frac{1}{2} + 0 \to \frac{1}{2} + 0$, if the product of the intrinsic parities of all the particles involved in the reaction is +1. On the other hand, if the product of the intrinsic parities is −1, then it is not hard to derive the following expression for the cross section of the reaction:

$$\sigma = \sigma_0(1 - \xi_\mu^{inc}\xi_\mu^0). \tag{6}$$

It can be verified that Eq. (5) is also valid in the case of scattering of polarized particles of spin $\frac{1}{2}$ by unpolarized particles with spin $\frac{1}{2}$ (for example, for nucleon-nucleon scattering).

In the laboratory coordinate system ($\mathbf{q} = 0$), and also in any other system moving relative to the laboratory system with a velocity lying in the plane of the scattering (for example in the center-of-mass system), the fourth component ξ_4^0 of the polarization vector is zero and the vector ξ^0 is directed along $[\mathbf{p} \times \mathbf{p}']$. Equation (5) then takes the following form:

$$\sigma = \sigma_0(1 + \xi^{inc}\xi^0).$$

Thus in the usual double-scattering experiment the asymmetry ϵ is equal to $\xi_\mu^0\xi_\mu^0$, i.e., to the square of the relativistically invariant polarization. A measurement of the asymmetry of the scattering of an arbitrarily polarized beam makes it possible to determine the component of the spatial part of the polarization vector normal to the plane of the scattering.

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ELECTROMAGNETIC STRUCTURE OF THE PROTON AND NEUTRON

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As is well known, the experimental results of the study of the charge and magnetic moment distribution of the nucleon have led to a sharp contradiction with the predictions of meson theory.[1] However, it seems to us that the difficulties are to a large extent caused by an insufficient understanding of the fact that the usual interpretation of the Hofstadter results is neither unique nor exact, but merely possible. The contradiction which supposedly exists between the law of distribution of the meson charge density according to Yukawa's theory $\sim e^{-\alpha r}/r^2$ and the experimentally-obtained law $\sim e^{-\beta r}$ is also without real significance as the range of applicability of these expressions is entirely different.

The total charge and magnetic moment density in the nucleon may be written as $\rho(r) = \rho_\pi(r) + \rho_k(r)$ and $m(r) = m_\pi(r) + m_k(r)$. Here $\rho_\pi(r)$ and $m_\pi(r)$ are the charge and magnetic moment densities of the meson cloud calculated with the Salzman theory [see Eq. (15) of reference 1] taking into account only the one-pion state. By $\rho_k(r)$ and $m_k(r)$ we denote the charge and magnetic moment densities of the core. Since at the present time very little is known about the two-, three-, and higher number pion states we shall consider them as part of the nucleon core.

With the specific form for the cut-off function

$$v(k) \equiv V(\omega) = \exp\{-\beta(\omega - 1)\}, \qquad \omega = \sqrt{k^2 + 1},$$

where $\omega_{max} = 1/\beta$ = cut-off frequency, we transform Salzman's expressions for $\rho_\pi(r)$ and $m_\pi(r)$ to the form

$$\rho_\pi(r) = e\mu^3\tau_3 \frac{2f^2}{\pi^3} e^{2\beta} r^2 \int_{\sqrt{r^2+\beta^2}}^{\infty} \frac{K_2^2(\rho)}{\rho^3\sqrt{\rho^2 - r^2}}\, d\rho,$$

$$\mathbf{m}_\pi(r) = e\mu^3 c\tau_3 \frac{f^2}{2(2\pi)^3} e^{2\beta}\, \mathbf{r} \times [\sigma \times \mathbf{r}] \left\{ \int_{\sqrt{r^2+\beta^2}}^{\infty} \frac{K_2(\rho)}{\rho\sqrt{\rho^2 - r^2}}\, d\rho \right\}^2,$$

where $K_2(\rho)$ is a Bessel function.

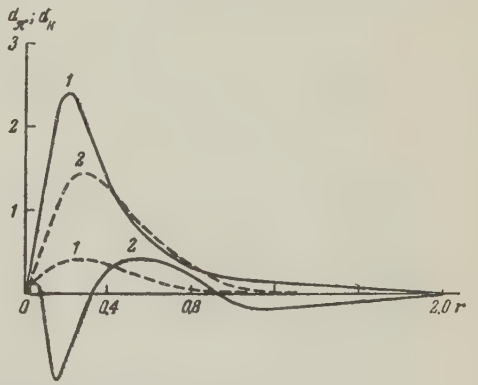
The results of the theory are not sensitive to the specific form of the cut-off. We choose $V(\omega)$ in a manner that gives simplest analytic expressions for $\rho_\pi(r)$ and $m_\pi(r)$, convenient for numerical calculations.

We choose $\beta = \hbar/7\mu_\pi c$ such that the calculated P-wave phase shift for pion-nucleon scattering is in as good agreement as possible with experiment in the low-energy region. Then we obtain for the electric and magnetic radii of the pion cloud $\langle r_e^2\rangle_\pi = 0.19\,(\hbar/\mu_\pi c)^2$ and $\langle r_m^2\rangle_\pi = 0.40\,(\hbar/\mu_\pi c)^2$, and for the charge and magnetic moment of the pion cloud $Q_\pi = 0.76\,e$ and $m_\pi = 1.25\,e\hbar/2Mc$.

The charge and magnetic moment distribution of the core will be taken in the form

$$\rho_k(r) = (Q_k/8\pi a^3)\, e^{-r/a}, \qquad m_k(r) = (m_k/8\pi a^3)\, e^{-r/a},$$

where Q_k and m_k are the charge and magnetic moment of the core. According to experiment the electric radius of the neutron $\langle r_e^2\rangle_n \approx 0$. When this is taken into account and when the anomalous magnetic moment of the nucleon is required to be equal to the experimental value $m_N = \tau_3 \times 1.85\,e\hbar/2Mc$ one finds, with $Q_k = (1+\tau_3)/2 - Q_\pi$, $\langle r_e^2\rangle_p = \langle r_m^2\rangle_n = \langle r_m^2\rangle_p = (0.7\,f)^2$ which is in good agreement with experiment. Here we took $a = \hbar/7\mu_\pi c \approx \hbar/Mc$. In the figure are shown the values



1 – proton structure, 2 – neutron structure; solid curves refer to the charge distribution $d(r) = 4\pi r^2 \rho(r)$ in the proton and neutron, dashed curves to the charge distribution $d_k(r) = 4\pi r^2 \rho_k(r)$ in their cores; r is in units of $\hbar/\mu_\pi c = 1.4 \times 10^{-13}$ cm; $d(r)$ and $d_k(r)$ are in units of $e\mu_\pi c/\hbar$.

of the charge in the proton and neutron and their cores.* Only for $r \gg \hbar/\mu_\pi c$, when

$$\rho(r) \sim e^{-2r}/r^2\sqrt{r}, \qquad m(r) \sim e^{-2r}/r^2,$$

does the density curve for the proton practically coincide with the curve in Hofstadter's work:[4] the density in the neutron, on the other hand, oscillates, and this explains the small electric radius of the neutron.

Consequently, the main results of Hofstadter can be brought into agreement with meson theory.

At that, the core is characterized by a small size $a \approx \hbar/Mc \ll \hbar/\mu_\pi c$.

*Our values for $\rho_\pi(r)$ are substantially different from those of reference 2; however, as was shown in reference 3, the results in reference 2 are in error.

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RESONANT SCATTERING OF GAMMA RAYS BY Ni^{60}

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WE observed the effect of resonant scattering of gamma rays by Ni^{60} nuclei by a procedure described by us earlier.[1,2] We used a gaseous $CoCl_2$ source. The gamma rays were detected with scintillation counters consisting of organic tolane crystals and FEU-33 photomultipliers. We recorded coincidences between the emitted cascade gamma quanta. The resolution of the coincident circuit was 2×10^{-9} sec.

Nickel and cobalt scatterers were placed alternately in front of one of the detectors. Within the γ-quanta emission-angle interval $180° > \varphi > 126°$ we observed for the nickel specimen an additional absorption of the 133-Mev gamma rays, the absorption being due to resonant scattering. No additional absorption was observed in the cobalt specimen.

We list the experimentally-determined cross sections (in cm^2) of resonant scattering for various angles φ:

φ	180°	150°	90°
$10^{25}\sigma_r =$	3.9 ± 1.2	1.7 ± 1.5	0 ± 1.2

These values agree, within the limits of error, with the σ_r vs. φ curve which we computed theoretically.[3]

The lifetime of the first excited level of Ni^{60} was found to be $\tau = (1.0 \pm 0.3) \times 10^{-13}$ sec (molecular bonds were taken into account in the calculations). This result is in good agreement with that of Metzger,[4] $\tau = (1.1 \pm 0.2) \times 10^{-12}$ sec, and agrees within the limits of error with the result of Alkhazov, Lemberg, et al.[5] obtained by the Coulomb excitation method, $\tau = 5.7 \times 10^{-13}$ sec with a 30% error.

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CERTAIN GAMMA TRANSITIONS IN I^{128} *AND IN NEODYMIUM ISOTOPES*

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USING a single-crystal luminescent spectrometer with NaI(Tl) crystal we investigated the gamma radiation produced in radiative capture of thermal neutrons in iodine and in neodymium isotopes. The measurement procedure was described earlier.[1,2]

I^{128}. The emission spectrum of this nucleus contained, in the energy region from 20 to 400 kev, gamma lines with energies 28 ± 2, 135 ± 3, and 158 ± 4 kev. Their respective intensities (percent per captured neutron) were 23 ± 6, 20 ± 4,

and 7.5 ± 1.5. The intensities of the 135 and 158-kev gamma quanta were obtained by resolving the summary photopeak, previously[1] ascribed to a single 135-kev gamma line. The 28-kev quanta are the characteristic K radiation of iodine, caused by internal conversion of the gamma rays of the reaction I(n, γ) by the electrons of the K shell of the atom. A comparison of the intensity of this radiation with the theoretical values of the internal-conversion coefficients[3] have made it possible to judge the character of the gamma transitions. The most probable multiplicities of the observed transitions, not in contradiction with the value of the intensity of the characteristic radiations, are E2 for the 135-kev transition and M2 for the 158-kev one. The considerable intensities of the gamma lines indicate that the corresponding transitions take place between low-lying excited levels of I^{128}.

Gamma-line energy (kev)	Gamma-line intensity for natural isotope mixture	Identification of gamma line	Gamma-line intensity (for isotopes)
182 ± 3	2.1 ± 0.4	Neodymium isotopes	
330 ± 10	23 ± 4	Sm^{150}	67
445 ± 10	25 ± 5	Sm^{150} Nd^{146}	40 >40
610 ± 10	20 ± 4	Sm^{150} Nd^{146}	16 ~100
695 ± 10	63 ± 10	Nd^{144}	85 ± 13
840 ± 10	15 ± 3	Nd^{144}	20 ± 4

Nd. The measurements were performed with an Nd_2O_3 target. The energy of the found gamma lines and their intensities, calculated for the natural mixture of isotopes (percent per capture neutron), are listed in the table. The identification of the gamma lines is made difficult by the large number of neodymium isotopes and possible contamination of the target by other rare-earth elements having large neutron capture cross sections. The chemical and mass-spectrometric analysis data indicate that the target contains samarium as an impurity. This makes it possible to attribute the 330-kev gamma line entirely and the 445 and 610-kev lines partially to radiation from the reaction Sm (n, γ).[1,4,5] It follows from the results of Sklyarevskiĭ et al.[6] that the gamma-ray spectra of radiative capture of thermal neutrons in Gd, Dy, and Er contain lines with energies ~180 and 80 — 90 kev, with an intensity ratio ranging from 2 to 5. In the present experiments we found no 80 — 90 kev peak of intensity comparable with that of the 182-kev peak. This means that the gamma quanta causing this peak must be due to the neodymium isotopes. The table contains an identification of the gamma lines, with account of the gamma-quanta intensities, the fractions of the captured neutrons belonging to the individual isotopes, and information on the levels of the neodymium isotopes, obtained by investigating the radioactive nuclei having neighboring values of Z.[7] The intensities of the gamma lines assigned to Sm^{150} were obtained by averaging the data of references 1, 4, and 5.

The 695- and 445-kev gamma rays are due to transitions from the first-excited to ground states of Nd^{144} and Nd^{146} respectively.[7] It is possible that there exist in Nd^{144} and Nd^{146} hitherto-unknown second excited states, with energies 1535 and 1055 kev, and transitions from these to the first excited levels give rise to emission of 840- and 610-kev gamma quanta. In this case the ratios of the energies of these states to the energies of the first levels, $E_2/E_1 = 2.2 - 2.4$, are characteristic of vibrational levels of spherical even-even nuclei.[8]

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THE UNIVERSAL FERMI INTERACTION AND ASTROPHYSICS

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J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1615-1616 (May, 1959)

THE hypothesis that there is a deep analogy between the various slow decay processes of elementary particles, both those involving leptons[1] and those not involving them,[2] has recently found a brilliant formulation in the theory of the universal (vector and axial-vector) interaction of Sudarshan and Marshak[3] and Feynman and Gell-Mann.[4] According to this theory the process of scattering of neutrinos by electrons is described by an interaction of the first order in the weak-interaction constant.[4,5] The experimental detection of such a process would be very desirable. But experimental work on the direct observation of the scattering of antineutrinos from a pile by electrons (that is, on the observation of ionization caused by the antineutrinos and not associated with inverse nuclear β processes) seems at present extremely difficult, though perhaps not altogether out of the question.

The purpose of the present note is to call attention to the fact that the existence of a first-order ν-e interaction would be capable of having macroscopic results. It follows from such an interaction that positron-electron annihilation can occur with the emission of neutrino-antineutrino pairs.* Therefore in electromagnetic processes it becomes possible for photon emission to be replaced by emission of a $\nu\tilde{\nu}$ pair (via a virtual e^+e^- pair). This general fundamental connection between electromagnetic phenomena and lepton processes follows directly from the universal Fermi interaction.

It is true that the emission of a $\nu\tilde{\nu}$ pair is extremely improbable relative to the emission of a photon, but the enormous penetrating power of neutrinos makes one think about the possibility of effects associated with the neutrino-electron interaction in large bodies at high temperature T. Let us examine the process of deceleration radiation of an electron with emission of a photon or a pair $\nu\tilde{\nu}$ in a collision of the electron with a nucleus A of charge Z:

$e + A \to e + A + \gamma$ (photon deceleration radiation)
$e + A \to e + A + \nu + \tilde{\nu}$ (lepton deceleration radiation)

Let us denote by α the ratio of the probabilities W_γ and $W_{\nu\tilde{\nu}}$ for the emission of a photon or a pair when an electron of energy E is deflected by the nucleus. Dimensional considerations lead us to set

$$\alpha = \frac{W_\gamma}{W_{\nu\tilde{\nu}}} \approx \frac{(e^2Z/\hbar c)^2 e^2/\hbar c}{(e^2Z/\hbar c)^2 G^2 (E/mc^2)^4},$$

where $G = gm^2c/h^3$ is the dimensionless weak-interaction constant, $g = 1.4 \times 10^{-49}$ erg cm^3 is the Fermi constant, and m is the mass of the electron.

It is clear that α is an enormous quantity at any temperatures encountered in astrophysics. Because of the difference of the penetrating power of photons and neutrinos, however, the emergence of a given energy (say $\sim kT$) from the stars into space in the form of photons will overall result from an enormous number of acts of photon deceleration radiation, incomparably larger than the number (~ 1) of acts of lepton deceleration radiation for the emergence of this same energy $\sim kT$ into space in the form of neutrinos. Therefore at a certain stage of the evolution of stars it may be that the energies sent into space in the forms of neutrinos and photons are comparable, in spite of the smallness of the ratio $W_{\nu\tilde{\nu}}/W_\gamma$ for each elementary act. Our attention is attracted by the strong temperature dependence of the probability of the process of lepton deceleration radiation, which is due to the dimensions of the Fermi constant. Besides this, with increase of A there is a decrease of the mean free path of photons, which leads to an increase of the importance of the neutrino process for the energy balance.

All of this leads to the idea that the process can become important in a stage of stellar evolution at which the temperature and the average Z considerably exceed the corresponding values for the sun. It is not hard to see that the mechanism of lepton deceleration ratiation plays practically no part in the energy balance of the sun ($kT \sim 1$ kev, $Z \sim 1$).

The mechanism of neutrino emission by stars proposed here is due to the neutrino-electron interaction and is fundamentally different from the process proposed by Gamow and Schoenberg,[6] which is due to nuclear (direct and inverse) β processes. The lepton deceleration radiation of the electron is a thresholdless process, whereas the Gamow-Schoenberg process is an effect with a definite threshold.

Recently G. M. Gandel'man and V. S. Pinaev have made a quantitative study of the astrophysical effect produced by the mechanism of lepton

deceleration radiation of the electron that has been described here. They have shown that in the temperature range $kT > 10$ kev and at densities $>10^5$ g/cm^3 the energy transferred from stars ($Z \approx 20$) is larger than the energy carried away by photons.

In conclusion I take great pleasure in thanking Ya. B. Zel'dovich, D. A. Frank-Kamenetskiĭ, and L. B. Okun' for critical comments and their interest in this work, and also G. M. Gandel'man and V. S. Pinaev, who have kindly informed me of the results of their work.

*In particular, the annihilation of orthopositronium with the emission of a $\nu\tilde{\nu}$ pair is less probable than three-photon annihilation by a factor of about 10^{15}. Because of the longitudinal character of the neutrino, parapositronium cannot undergo annihilation with the emission of a neutrino and antineutrino.

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